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SUPERSYMMETRIC RACAH BASIS, FAMILY OF INFINITE-DIMENSIONAL SUPERALGEBRAS, SU(∞ + 1 | ∞) AND RELATED 2D MODELS

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The irreducible Racah basis for SU(N + 1 | N) is introduced. An analytic continuation with respect to N leads to infinite-dimensional superalgebras $\underline{\text{su}}(\nu + 1 \mid \nu)$. The large-$\nu$ limit su(∞ + 1 | ∞) is calculated. A higher spin Sugawara construction leading to generalizations of the Virasoro algebra with infinite tower of higher spin currents is proposed, and the related WZNW and Toda models as well as the possible applications in string theory are discussed.

1. Introduction

Recently a nice continuously-parametric family[a] of infinite-dimensional (inf-dim) associative Lie algebras, namely, quantum operatorial algebras of quantum systems with spherical S^2 or hyperbolical $S^{1,1}$ phase space, has attracted some attention in various contexts.[1-5] In the classical limit these algebras are contracted to the algebras of symplectic (area-preserving) diffeomorphisms of S^2 or $S^{1,1}$, which have been studied extensively in the context of relativistic membranes.[6,7] They can be viewed as natural generalizations of the classical Lie algebras.

In the present letter we will discuss the supersymmetric versions of the above algebras and their applications in 2D QFT. Particularly we obtain complete manifest expressions for their structure constants. Our strategy will be the following. We start with the fin-dim classical superalgebras su(N + 1 | N) and introduce for them a special SU(2)-irreducible basis supersymmetric generalization of the SU(2)-irreducible Racah basis for su(N), and calculate the su(N + 1 | N) structure constants in this basis. Then, after a redefinition of the generators representing the structure constants as polynomial functions of N, we perform an analytic continuation from integers N to all non-integer real ν and obtain a continuous family of inf-dim superalgebras $\underline{\text{su}}(\nu + 1 \mid \nu)$. Then we are able to pass to a limit $\nu \to \infty$ and calculate the structure constants of the resulting superalgebra su(∞ + 1 | ∞), based on the general properties of the 6j-symbols. The resulting superalgebra su(∞ + 1 | ∞) may be viewed as an algebra of orthosymplectic superdiffeomorphisms of the coset supermanifold SU(2 | 1)/U(1 | 1), and $\underline{\text{su}}(\nu + 1 \mid \nu)$ as its quantum deformations.

[a] Note that one member of this family (as well as its supersymmetric extensions) came from Refs. 8 and 9 in the context of gauge theories of higher spin fields.

Among the various possible applications we propose a systematic way to construct higher spin generalizations of the Virasoro algebra involving an infinite tower of higher spin generators based on higher spin Sugawara construction for $\underline{su}(\nu)$ and $\underline{su}(\nu+1 \mid \nu)$ and their large-ν limits. A crucial point here is the existence of an infinite tower of independent Casimir invariants for these inf-dim (super)algebras. It allows us to build up an infinite number of the Casimir currents from the affine $\underline{su}(\nu)$ chiral currents. In principle, WZNW models on the inf-dim group manifolds may serve as origins for such Kac-Moody currents. Special reduction by means of certain constraints may give rise to the Toda models with an infinite number of fields and the above described Casimir algebra in the place of W-algebra with a finite number of the higher spin currents. Possible applications in string theory are discussed in conclusion.

2. The Algebras SU(N), $\underline{\mathrm{SU}}(\nu)$, and SU($\infty$)

Let T_i be some basis in sl_2 and C be the quadratic sl_2 Casimir element. The universal enveloping algebra $\mathrm{U}(\mathrm{sl}_2)$ is an associative algebra with a unit 1 and generating elements T_i obeying sl_2 commutation relations. $\mathrm{U}(\mathrm{sl}_2)$ contains a center $Z(\mathrm{sl}_2)$ generated by C. By considering a family of ideals $\tau(\lambda) = (C - \lambda 1)\, \mathrm{U}(\mathrm{sl}_2)$ (λ is an arbitrary number), one can define a family of factor-algebras $\mathrm{U}(\mathrm{sl}_2 \mid \lambda) = \mathrm{U}(\mathrm{sl}_2)/\tau(\lambda)$. The commutator AB-BA transforms all the associative algebras into Lie algebras $[\mathrm{U}(\mathrm{sl}_2 \mid \lambda)]$.

As has been shown in Refs. 2 and 3 these algebras are pairwise non-isomorphic at different values of the Casimir element λ. The algebras $\mathrm{U}(\lambda)$ and $L(\lambda) = [\mathrm{U}(\lambda)]/$ (one-dimensional center generated by 1) are simple for the general position values of λ. It is a remarkable fact that at the exceptional points $\lambda = -(N^2-1)/4$ with integer N there exists an ideal χ_N in $\mathrm{U}(\lambda)$ such that the following isomorphism takes place

$$\mathrm{Mat}_N \simeq \mathrm{U}(-(N^2-1)/4)/\chi_N\,, \tag{1}$$

and for the Lie algebra,

$$\mathrm{sl}_N \simeq L(-(N^2-1)/4)/\chi_N \tag{2}$$

(Mat_N is an algebra of all $N \times N$ matrices).

Equations (1) and (2) have been obtained in Ref. 5 by calculating the invariant symmetric bilinear forms. It turned out that at the exceptional point the bilinear form becomes degenerate and χ_N is just its null-space. Taking into account (2), we also introduce the notation $\underline{\mathrm{sl}}(\nu) = L\,(-(\nu^2-1)/4)$, where $0 \le \nu < \infty$ is a continuous parameter (ν and $-\nu$ define the same eigenvalue λ of C). Then $\mathrm{sl}(N) = \mathrm{sl}(N)/\chi_N$, and $\underline{\mathrm{sl}}(\nu)$ can be called analytic continuation of $\mathrm{sl}(N)$ with respect to the parameter N. Correspondingly we also use the notations $\underline{A}_\nu$, and $\underline{\mathrm{su}}(\nu)$ and $\underline{\mathrm{sl}}(\nu; \mathrm{R})$ for compact and non-compact real forms ($0 \le \nu < \infty$).

The composition law in $\underline{\mathrm{sl}}(\nu)$ may be obtained from the composition law of $\mathrm{sl}(N)$ in the Racah basis.[1] Let E_{ij}, $i, j = 1,\dots, N$, be the usual basis in Mat_N with the composition law $E_{ij}E_{kl} = \delta_{jk}E_{il}$. It was Racah[10] (see also Ref. 11) who introduced in

Mat_N a new basis

$$T^s_m = \sqrt{\frac{2s+1}{N}} \sum_{m',m''} C^{jsj}_{m'mm''} E_{j-m''+1, j-m'+1} , \tag{3}$$

$s = 0, 1, \ldots, N-1$, $|m| \le s$, $j = (N-1)/2$, where C are the usual Clebsh-Gordan coefficients (all our notations and conventions concerning the angular momentum theory follow Ref. 11). Calculating the commutator of two matrices (3), one arrives at the Racah commutation relations for gl_N:

$$[T^s_m, T^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''|N)\, C^{ss's''}_{mm'm''}\, T^{s''}_{m''} , \tag{4}$$

$$f(s,s',s''|N) = (1-(-1)^{s+s'-s''})(-1)^{s''+N-1} \times \sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix}, \quad j = \frac{N-1}{2} , \tag{5}$$

with the $6j$-symbols appearing in the structure constants. The crucial feature of the Racah basis is that all the generators T^s_m with fixed $s \le N-1$ are transformed under irreducible spin-s representation of the total angular momentum subgroup generated by sl_2 generators T^1_m. The trivial representation $T^0_0 \sim 1$ forms a center in gl_N and $\mathrm{sl}_N = \mathrm{gl}_N/\mathbb{C}1$. Compact real form su($N$) is extracted by means of the Hermitian conjugation $(T^s_m)^\dagger = (-1)^m T^s_{-m}$. The Racah basis in SU($N$) was applied in the studies of the relativistic spherical membranes in Ref. 6. The invariant bilinear form in the Racah basis reads as follows,

$$(T^s_m, T^{s'}_{m'}) = \mathrm{tr}\,(T^s_m T^{s'}_{m'}) = (-1)^m\, \delta^{s,s'} \delta_{m,-m'} , \tag{6}$$

where tr is the usual matrix trace. After redefinition of the generators

$$\tilde{T}^s_m = \sqrt{\frac{(N+s)!}{(2s+1)(N-s-1)!}}\; T^s_m , \tag{7}$$

the structure constants in Eqs. (4) and (5) become polynomial functions on N and now we are able to make an analytic continuation from integers N to all real ν.[1] The resulting commutation relations have a form

$$[\tilde{T}^s_m, \tilde{T}^{s'}_{m'}] = \sum_{s'',m''} (1-(-1)^{s+s'-s''})\, f^{ss's''}_{000}(\nu)\, C^{ss's''}_{mm'm''}\, \tilde{T}^{s''}_{m''} , \tag{8}$$

where the reduced structure constants are expressed through quantities given in (A1) ($k = k' = k'' = 0$ in this case). They are the generalized $6j$-symbols.

In Eq. (8), apart from the analytic continuation from integer N to real ν, it is necessary also to abolish the restrictions $s \le N-1$ and now all higher spins up to infinity are involved in the inf-dim algebra. The generalized $6j$-symbols $f^{ss's''}_{000}(-\nu) = f^{ss's''}_{000}(\nu)$ (due to (A3)) are in fact functions of an eigenvalue $\lambda = -(\nu^2 - 1)/4$ of the sl_2 Casimir operator, and consequently Eq. (8) gives the commutation relations in $\underline{\mathrm{sl}}(\nu)$ (and in $\underline{\mathrm{su}}(\nu)$).[1]

The invariant symmetric bilinear form in $\underline{\mathrm{sl}}(\nu)$ is given by

$$(\tilde{T}^{s}_{m}, \tilde{T}^{s'}_{m'}) = \mathrm{tr}\,(\tilde{T}^{s}_{m} \cdot \tilde{T}^{s'}_{m'}) = \frac{(-1)^{m}}{(2s+1)}\, \delta^{s,s'}\delta_{m,-m'} \prod_{p=-s}^{s} (\nu+p)\,. \tag{9}$$

As we have already noticed, at the exceptional integer point (when $\nu = N$ is integer) the bilinear form (9) becomes degenerate and its null-space is just the ideal χ_N formed by T^s_m with $s \geq N$.[5] Restricting on the factor-algebra by modulo χ_N, the tr becomes the usual matrix trace.

Further one can pass to a limit $\nu \to \infty$. After redefinition $\tilde{\tilde{T}}^{s}_{m} = -\nu^{-s+1}\, \tilde{T}^{s}_{m}$, a resulting algebra sl(∞) (su(∞))[b] is given by (4) with the reduced structure constants obtained in the limit $\nu \to \infty$,

$$f(s,s',s''|\infty) = (1-(-1)^{s+s'-s''}) \times \sqrt{s\,s'(s+s'-s'')\,(s+s'+s''+1)}\; C^{s-1/2,\, s'-1/2,\, s''}_{1/2,-1/2,0} \tag{10}$$

(see (8), (A7), and note that $C^{ss's''}_{000} = 0$ for odd $s + s' + s''$). This limit has been evaluated in Ref. 6.

The generators T^s_m with odd spins s form a subalgebra in $\underline{\mathrm{sl}}(\nu)$ owing to the presence of the factor $1-(-1)^{s+s'-s''}$ in Eqs. (4) and (8). We denote it by $\underline{\mathrm{spo}}(\nu)$ because of the following isomorphisms, so(2n + 1; C) ≃ $\underline{\mathrm{spo}}(2n+1;\mathrm{C})/\chi^{\mathrm{odd}}_{2n+1}$, sp(2$n$; C) ≃ $\underline{\mathrm{spo}}(2n;\mathrm{C})/\chi^{\mathrm{odd}}_{2n}$ (χ^{odd}_{N} is formed by T^s_m with $s \geq N$ and s odd), and similarly for the compact and non-compact real forms. The algebra so(N) (sp(N)) is an odd-spin subalgebra of sl(N) generated by the Racah generators T^s_m with s = 1, 3,..., $N-2(N-1)$ for odd(even) N. In this way the classical series B_N and C_N are combined into one continuous family of algebras $\underline{BC}_\nu$ and can be obtained as factor-algebras in odd and even exceptional integer-ν points. Note also that in this construction the series B_N and C_N have a joint large-N limit spo(∞) ⊂ su(∞).

The above algebraic construction has a number of physical and geometrical realizations.

(i) Compact $\underline{\mathrm{su}}(\nu)$ and non-compact sl(ν; R) real algebras are the quantum operatorial Lie algebras on the sphere S^2 and the hyperboloid $S^{1,1}$ served as phase spaces.[1-5] The parameter ν in this realization is an inverse quantum deformation parameter ($\nu \sim 1/\hbar$). In the classical limit $\nu \to \infty$ ($\hbar \to 0$), one obtains an algebra SU(∞) of area-preserving (symplectic) diffeomorphisms of S^2 (or $S^{1,1}$ in non-compact case).

(ii) The algebra U(sl$_2$) can be embedded into the Weyl algebra generated by four (twistor-like) Heisenberg generating elements a_α, b_α, where α = 1, 2. It gives the oscillator-like realization of $\underline{\mathrm{sl}}(\nu)$ (see Appendix C of the last reference in Ref. 4).

(iii) $\underline{\mathrm{sl}}(\nu)$ can be realized in terms of the differential operators on the circle[4,1] S^1:

[b] In the literature, various algebras are denoted by SU(∞). In this paper SU(∞) is just the algebra of symplectic diffeomorphisms of the sphere S^2.

$$T_m^{(\nu)s} = \sum_{k=0}^{s} \left\{ \prod_{p=k+1}^{s} \frac{p(p+m)(p+\nu)}{(p+s)(p-s-1)} \right\} z^{m+k} \frac{d^k}{dz^k} . \tag{11}$$

(iv) A rather unusual realization was obtained in Ref. 5 in terms of polynomials of two spinorial generating elements $q_\alpha (\alpha = 1,2)$ and the Klein operator $Q(Q^2 = 1)$, $\{Q, q_\alpha\} = 0$.
(v) In Ref. 2 the algebras $\mathrm{U}(\mathrm{sl}_2 \mid \lambda)$ and $\underline{\mathrm{sl}}(\nu)$ were realized as algebras of infinite matrices.

With the Lie algebras $\underline{\mathrm{su}}(\nu)$ and $\underline{\mathrm{sl}}(\nu)$ one can associate the corresponding infinite-dimensional groups. Certainly in the inf-dim case the correspondence among algebras and groups is not so direct, but in this case the definition of the groups may seemingly be given correctly owing to the geometrical realization as quantized symplectic diffeomorphisms. Quantum mechanically, the algebra $\underline{\mathrm{su}}(\nu)$ is realized as an algebra of (anti)Hermitian operators in the Hilbert space of states of the quantum system with S^2 as a phase space ($1/\nu$ is a quantum deformation parameter). Then the corresponding group $\underline{\mathrm{SU}}(\nu)$ can be interpreted as a group of unitary operators in this space, i.e., formally $\underline{\mathrm{SU}}(\nu)$ = exp $\underline{\mathrm{su}}(\nu)$, and similarly for the non-compact case of hyperboloid. In the exceptional points the Hilbert space has an invariant fin-dim subspace and the restriction of operators on this subspace is equivalent to the transition to a factor-group $\mathrm{SU}(N) \simeq \underline{\mathrm{SU}}(N)/\exp \chi_N$, where exp χ_N is a normal subgroup corresponding to the ideal χ_N in the algebra. Such special quantizations on the sphere were considered firstly by Berezin[12] (see also Refs. 5 and 1). The group of symplectic diffeomorphisms SDIFF(S^2) is a classical limit of the groups $\underline{\mathrm{SU}}(\nu)$ at $\nu \to \infty$ ($\hbar \to 0$).

3. Supersymmetric Racah Basis in SU($N+1 \mid N$)

Now we are going to construct manifestly a supersymmetric extension of the above construction. First of all we will build up a superanalog of the Racah basis. As a starting point let us take Mat_{2N+1} with some integer N and the usual basis $E_{I,J}$, $I, J = 1,\dots, 2N+1$, $E_{I,J}E_{K,L} = \delta_{J,K}E_{I,L}$. Let us divide the set $E_{I,J}$ into four subsets:

$$E_{\underline{i},\underline{j}} = E_{i,j}\,(i, j = 1,\dots, N+1) , \tag{12}$$

$$E_{\bar{i},\bar{j}} = E_{i+N+1,\, j+N+1}\,(i, j = 1,\dots, N) , \tag{13}$$

$$E_{\underline{i},\bar{j}} = E_{i,\, j+N+1}\,(i = 1,\dots, N+1, j = 1,\dots, N) , \tag{14}$$

$$E_{\bar{i},\underline{j}} = E_{i+N+1,j}\,(i = 1,\dots, N, j = 1,\dots, N+1) . \tag{15}$$

Evidently, $E_{\underline{i},\underline{j}}$ and $E_{\bar{i},\bar{j}}$ form a basis in the subalgebra $\mathrm{Mat}_{N+1} \oplus \mathrm{Mat}_N \subset \mathrm{Mat}_{2N+1}$. In each summand we can introduce a new basis:

$$T_m^s = \sqrt{\frac{2s+1}{N+1}} \sum_{m',m''} C_{m'mm''}^{N/2\, s\, N/2} E_{\underline{N/2-m''+1}\,,\,\underline{N/2-m'+1}} , \; (s = 0, 1,\dots, N) , \tag{16}$$

$$U^s_m = \sqrt{\frac{2s+1}{N}} \sum_{m',m''} C^{(N-1)/2\, s\, (N-1)/2}_{m'mm''} E_{\overline{(N-1)/2-m''+1},\, \overline{(N-1)/2-m'+1}}, \quad (s = 0, 1, \ldots, N-1), \tag{17}$$

$$\overline{Q}^s_m = \sqrt{\frac{2s+1}{N+1}} \sum_{m',m''} C^{(N-1)/2\, s\, N/2}_{m'mm''} E_{\underline{N/2-m''+1},\, \overline{(N-1)/2-m'+1}}, \quad \left(s = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}\right), \tag{18}$$

$$Q^s_m = \sqrt{\frac{2s+1}{N}} \sum_{m',m''} C^{N/2\, s\, (N-1)/2}_{m'mm''} E_{\overline{(N-1)/2-m''+1},\, \underline{N/2-m'+1}}, \quad \left(s = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}\right), \tag{19}$$

and $|m| \le s$ for all the generators. Now let us introduce a $\mathbb{Z}_2$-grading by means of the Grassmann parity function $P(T)=P(U)=0$, $P(Q)=P(\overline{Q})=1$, i.e., T and U are the Bose generators (in fact, the Racah basis in $\mathrm{Mat}_{N+1} \oplus \mathrm{Mat}_N$) and Q and $\overline{Q}$ – the Fermi ones. Then Eqs. (16) – (19) define a Racah basis in Mat_{2N+1} viewed as an algebra of the supermatrices $\mathrm{Mat}_{N+1|N}$. Introducing a supercommutator

$$[A, B\} = AB - (-1)^{P(A)\, P(B)} BA \tag{20}$$

for any two supermatrices A and B (with defined parity) from $\mathrm{Mat}_{N+1|N}$, one transforms $\mathrm{Mat}_{N+1|N}$ into the Lie superalgebra $\mathrm{gl}(N+1\,|\,N; \mathbb{C})$ with the even part $\mathrm{gl}(N+1; \mathbb{C}) \oplus \mathrm{gl}(N; \mathbb{C})$ formed by T and U respectively, and the odd part formed by Q and $\overline{Q}$.

The non-zero supercommutation relations in $\mathrm{gl}(N+1\,|\,N)$ in the Racah basis take the following form,

$$[T^s_m, T^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''\,|\,N+1)\, C^{ss's''}_{mm'm''}\, T^{s''}_{m''}\,, \tag{21a}$$

$$[U^s_m, U^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''\,|N)\, C^{ss's''}_{mm'm''}\, U^{s''}_{m''}\,, \tag{21b}$$

$$\{Q^s_m, \overline{Q}^{s'}_{m'}\} = \sum_{s'',m''} C^{ss's''}_{mm'm''} \left(g_1(s,s',s''|N)\, T^{s''}_{m''} + g_2(s,s',s''|N)\, U^{s''}_{m''}\right), \tag{21c}$$

$$[T^s_m, \overline{Q}^{s'}_{m'}] = \sum_{s'',m''} h_1(s,s',s''\,|\,N)\, C^{ss's''}_{mm'm''}\, \overline{Q}^{s''}_{m''}\,, \tag{21d}$$

$$[U^s_m, \overline{Q}^{s'}_{m'}] = \sum_{s'',m''} h_2(s,s',s''\,|\,N)\, C^{ss's''}_{mm'm''}\, \overline{Q}^{s''}_{m''}\,, \tag{21e}$$

$$[T^s_m, Q^{s'}_{m'}] = \sum_{s'',m''} (-1)^{s+s'-s''-1} h_1(s,s',s''\,|\,N)\, C^{ss's''}_{mm'm''}\, Q^{s''}_{m''}\,, \tag{21f}$$

$$[U^s_m, Q^{s'}_{m'}] = \sum_{s'',m''} (-1)^{s+s'-s''-1} h_2(s,s',s''\,|\,N)\, C^{ss's''}_{mm'm''}\, Q^{s''}_{m''}\,, \tag{21g}$$

where $f(s, s', s''\,|\,N)$ are given by Eq. (5) and the other reduced structure constants are expressed through the $6j$-symbols as follows,

$$g_1(s,s',s''|N) = (-1)^{s+s'+N-1} \sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ \frac{N}{2} & \frac{N}{2} & \frac{N-1}{2} \end{Bmatrix}, \tag{22a}$$

$$g_2(s,s',s''|N)=(-1)^{s''+N}\sqrt{(2s+1)(2s'+1)}\begin{Bmatrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N}{2}\end{Bmatrix}, \tag{22b}$$

$$h_1(s,s',s''|N)=(-1)^{s''+N-1/2}\sqrt{(2s+1)(2s'+1)}\begin{Bmatrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N}{2} & \frac{N}{2}\end{Bmatrix}, \tag{22c}$$

$$h_2(s,s',s''|N)=(-1)^{s+s'+N+1/2}\sqrt{(2s+1)(2s'+1)}\begin{Bmatrix} s & s' & s'' \\ \frac{N}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{Bmatrix}. \tag{22d}$$

To calculate the structure constants of the superalgebra gl(N + 1 | N) in the Racah basis we have used the inverse transformation for Eqs. (16)–(19)

$$E_{\underline{N/2-m''+1},\underline{N/2-m'+1}}=\sum_{s,m}\sqrt{\frac{2s+1}{N+1}}\,C^{N/2\; sN/2}_{m'mm''}\,T^s_m\,, \tag{23a}$$

$$E_{\overline{(N-1)/2-m''+1},\overline{(N-1)/2-m'+1}}=\sum_{s,m}\sqrt{\frac{2s+1}{N}}\,C^{(N-1)/2\; s(N-1)/2}_{m'mm''}\,U^s_m\,, \tag{23b}$$

$$E_{\underline{N/2-m''+1},\overline{(N-1)/2-m'+1}}=\sum_{s,m}\sqrt{\frac{2s+1}{N+1}}\,C^{(N-1)/2\; sN/2}_{m'mm''}\,\overline{Q}^s_m\,, \tag{23c}$$

$$E_{\overline{(N-1)/2-m''+1},\underline{N/2-m'+1}}=\sum_{s,m}\sqrt{\frac{2s+1}{N}}\,C^{N/2\; s(N-1)/2}_{m'mm''}\,Q^s_m\,, \tag{23d}$$

and allowed for the intertwining formula for three Clebsh-Gordan coefficients, (A9).

The compact real form U(N + 1 | N) is extracted by means of the Hermitian conjugation of the supermatrices,

$$\begin{aligned}(T^s_m)^\dagger &= (-1)^m T^s_{-m}\,, \quad (U^s_m)^\dagger = (-1)^m U^s_{-m}\,,\\ (Q^s_m)^\dagger &= (-1)^{m-1/2}\,\overline{Q}^s_{-m}\,, \quad (\overline{Q}^s_m)^\dagger = (-1)^{m+1/2} Q^s_{-m}\,.\end{aligned} \tag{24}$$

The unit matrix $\mathbb{1}=\sqrt{N+1}\,T^0_0+\sqrt{N}\,U^0_0$ generates a center in gl($N+1\,|N$; ℂ) and one has sl($N+1\,|\,N$; ℂ) ≃ gl($N+1\,|\,N$; ℂ)/ℂ𝟙. The generators

$$L_{\pm1}=\sqrt{\frac{N(N+1)}{6}}\,(\sqrt{N+2}\,T^1_{\pm1}+\sqrt{N-1}\,U^1_{\pm1})\,,$$

$$L_0=\sqrt{\frac{N(N+1)}{12}}\,(\sqrt{N+2}\,T^1_0+\sqrt{N-1}\,U^1_0)$$

form a basis in the total sl_2 subalgebra (or in SU(2) when (24) is supposed to be satisfied and $A^\dagger=-A$ for any $A\in$ su(N + 1 | N). All the other generators are transformed under irreducible representations of this "total angular momentum subalgebra." This is a peculiar feature of the Racah basis (16) – (19). The generators L_m accompanied by $Q_m^{1/2}$, $\overline{Q}_m^{1/2}$ and $U = \sqrt{N}\,T_0^0 + \sqrt{N+1}\,U_0^0$ (str $U = 0$) form a basis in the total superalgebra sl(2 | 1; C) (or su(2 | 1)).

Under an invariant symmetric bilinear form $(A, B) = \mathrm{str}\,(AB)$, where str is the matrix supertrace, the Racah supergenerators are normalized as follows (all the other pairings are equal to zero),

$$(T_m^s, T_{m'}^{s'}) = (U_m^s, U_{m'}^{s'}) = (-1)^m \delta^{s,s'} \delta_{m,-m'} \,,$$
$$(Q_m^s, \overline{Q}_{m'}^{s'}) = (-1)^{m-1/2} \delta^{s,s'} \delta_{m,-m'}, (\overline{Q}_m^s, Q_{m'}^{s'}) = (-1)^{m+1/2} \delta^{s,s'} \delta_{m,-m'} \,. \tag{25}$$

4. The Superalgebras su(ν + 1) and su(∞ + 1 | ∞)

Now, similar to the purely bosonic case, we can perform an analytic continuation of sl(N + 1 | N) with respect to N and obtain as a result a continuously parametric family sl(ν + 1 | ν) of infinite-dimensional superalgebras. Precisely, after redefinition of the Racah generators

$$\tilde{T}_m^s = \sqrt{\frac{(N+s+1)!}{(2s+1)!\,(N-s)!}}\,T_m^s,\quad \tilde{U}_m^s = \sqrt{\frac{(N+s)!}{(2s+1)!\,(N-s-1)!}}\,U_m^s$$
$$\tilde{Q}_m^s = \sqrt{\frac{\left(N+s+\frac{1}{2}\right)!}{(2s+1)!\left(N-s-\frac{1}{2}\right)!}}\,Q_m^s,\quad \tilde{\overline{Q}}_m^s = \sqrt{\frac{\left(N+s+\frac{1}{2}\right)!}{(2s+1)!\left(N-s-\frac{1}{2}\right)!}}\,\overline{Q}_m^s \,, \tag{26}$$

abolishing the restrictions $s \le N$, $s \le N - 1/2$, $s \le N - 1$ for $\tilde{T}_m^s$, $\tilde{\overline{Q}}_m^s$, $\tilde{U}_m^s$ respectively and with analytic continuation from integer N to real $\nu \in \mathbb{R}$, $\nu \ge 0$ we arrive at the inf-dim superalgebras sl(ν + 1 | ν) with supercommutation relations of the form (21), where the reduced structure constants are now given by (N is replaced by ν):

$$f(s, s', s''|\nu) = (1 - (-1)^{s+s'-s''}) f_{000}^{ss's''}(\nu)\,, \tag{27a}$$
$$g_1(s, s', s''|\nu) = (-1)^{s+s'-s''+1} f_{1/2\,-1/2\,0}^{ss's''}(\nu+1)\,, \tag{27b}$$
$$g_2(s, s', s''|\nu) = -f_{-1/2\,1/2\,0}^{ss's''}(\nu)\,, \tag{27c}$$
$$h_1(s, s', s''|\nu) = -f_{0\,1/2\,1/2}^{ss's''}(\nu)\,, \tag{27d}$$
$$h_2(s, s', s''|\nu) = (-1)^{s+s'-s''+1} f_{0\,-1/2\,-1/2}^{ss's''}(\nu+1)\,, \tag{27e}$$

(see Appendix (A1)).

An invariant bilinear form for sl(ν + 1 | ν) has the form

$$(\tilde{T}_m^s, \tilde{T}_{m'}^{s'}) = \frac{(-1)^m}{2s+1} \delta^{s,s'} \delta_{m,-m'} \prod_{p=-s}^{s} (p+\nu+1)\,, \tag{28a}$$

$$(\tilde{U}^{s}_{m},\tilde{U}^{s'}_{m'}) = \frac{(-1)^{m}}{2s+1}\,\delta^{s,s'}\delta_{m,-m'}\prod_{p=-s}^{s}(p+\nu)\,, \tag{28b}$$

$$(\tilde{Q}^{s}_{m},\bar{\tilde{Q}}^{s'}_{m'}) = \frac{(-1)^{m-1/2}}{2s+1}\,\delta^{s,s'}\delta_{m,-m'}\prod_{p=-s}^{s}\left(p+\nu+\frac{1}{2}\right), \tag{28c}$$

$$(\bar{\tilde{Q}}^{s}_{m},\tilde{Q}^{s'}_{m'}) = \frac{(-1)^{m+1/2}}{2s+1}\,\delta^{s,s'}\delta_{m,-m'}\prod_{p=-s}^{s}\left(p+\nu+\frac{1}{2}\right). \tag{28d}$$

This bilinear form is non-degenerate for all non-integer values of ν. However, at the integer point $\nu = N$ it becomes degenerate. The corresponding null-space is formed by the generators $\tilde{T}^{s}_{m}$, $\tilde{Q}^{s-1/2}_{m}$, $\bar{\tilde{Q}}^{s-1/2}_{m}$, $\tilde{U}^{s-1}_{m}$ with $s = N+1, N+2, \ldots$.

Factoring out this null-space (which is an ideal superanalog of χ_N) we come back to the fin-dim superalgebras sl$(N+1\,|\,N)$ and su$(N+1\,|\,N)$. This result is in agreement with that of Ref. 5 obtained in another way.

Now having a continuous family of superalgebras at our disposal, we can pass to a large-ν limit and obtain a supersymmetric version of SU(∞), namely SU$(\infty + 1\,|\,\infty)$.[c] To evaluate the limit, first let us make a redefinition of the generators:

$$\tilde{L}^{s}_{m} = -\,\nu^{s+1}(\tilde{T}^{s}_{m}+\tilde{U}^{s}_{m}),\quad V^{s}_{m} = -\,\nu^{-s}(\tilde{T}^{s}_{m}-\tilde{U}^{s}_{m})\,, \tag{29a}$$

$$\tilde{T}^{s}_{m} = -\frac{1}{2}\,\nu^{s-1}(\tilde{L}^{s}_{m}+\nu V^{s}_{m}),\quad \tilde{U}^{s}_{m} = -\,\nu^{s-1}\,\frac{1}{2}(\tilde{L}^{s}_{m}-\nu V^{s}_{m})\,, \tag{29b}$$

$$Q^{s}_{m} = \nu^{s-1/2}\tilde{Q}^{s}_{m}\,,\quad \bar{Q}^{s}_{m} = \nu^{s-1/2}\bar{\tilde{Q}}^{s}_{m}\,. \tag{29c}$$

Then after calculating the commutation relations in the new basis (see Eqs. (21) and (22)), taking a limit $\nu \to \infty$ (with the help of (A7)), and again redefining the generators,

$$L^{s}_{m} = \tilde{L}^{s}_{m} - \frac{1}{2}(s-1)\,V^{s}_{m}\,, \tag{29d}$$

we arrive at the following commutation relations for su$(\infty + 1\,|\,\infty)$,

$$[L^{s}_{m},L^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''\,|\,\infty)\,C^{ss's''}_{mm'm''}\,L^{s''}_{m''}\,, \tag{30a}$$

$$[L^{s}_{m},V^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''\,|\,\infty)\,C^{ss's''}_{mm'm''}\,V^{s''}_{m''}\,, \tag{30b}$$

$$[V^{s}_{m},V^{s'}_{m'}] = 0\,, \tag{30c}$$

$$[V^{s}_{m},\bar{Q}^{s'}_{m'}] = \sum_{s'',m''} C^{ss's''}_{0\,1/2\,1/2}\,C^{ss's''}_{mm'm''}\,\bar{Q}^{s''}_{m''}\,, \tag{30d}$$

$$[V^{s}_{m},Q^{s'}_{m'}] = -\sum_{s'',m''} C^{ss's''}_{0\,-1/2\,-1/2}\,C^{ss's''}_{mm'm''}\,Q^{s''}_{m''}\,, \tag{30e}$$

[c] We have also obtained the N-extended supersymmetric versions of SU(∞) by quite a different approach (calculating the Poisson superbrackets on the twistor superspace, to be published).

$$[L^s_m, \overline{Q}^{s'}_{m'}] = 2 \sum_{s'',m''} f^{ss's''}_{0\,1/2\,1/2}\left(1, \frac{1}{2}\right) C^{ss's''}_{mm'm''}\, \overline{Q}^{s''}_{m''}\,, \tag{30f}$$

$$[L^s_m, Q^{s'}_{m'}] = 2 \sum_{s'',m''} f^{ss's''}_{0\,-1/2\,-1/2}\left(1, \frac{1}{2}\right) C^{ss's''}_{mm'm''}\, Q^{s''}_{m''}\,, \tag{30g}$$

$$\{Q^s_m, \overline{Q}^{s'}_{m'}\} = \sum_{s'',m''} C^{ss's''}_{mm'm''} \left\{ C^{ss's''}_{-1/2\,1/2\,0}\, L^{s''}_{m''} - 2 f^{ss's''}_{-1/2\;1/2\;0}\left(\frac{1}{2}, \frac{1}{2}\right) V^{s''}_{m''} \right\}, \tag{30h}$$

where the reduced structure constants read

$$f^{ss's''}_{cc'c''}(p, p') = \frac{1}{2}\sqrt{(s-c)(s'+c')(s+s'-s'')(s+s'+s''+1)} \times C^{s-1/2,\, s'-1/2,\, s''}_{c+1/2,\, c'-1/2,\, c''} + \frac{1}{2}\sqrt{(s+c)(s'-c')(s+s'-s'')(s+s'+s''+1)} \times C^{s-1/2,\, s'-1/2,\, s''}_{c-1/2,\, c'+1/2,\, c''} + [(p'-s')c-(p-s)c']C^{ss's''}_{cc'c''} \tag{31}$$

(in particular, $f(s, s', s''|\infty) = f^{ss's''}_{000}(p, p')$ does not depend on p, p'). Such quantities originally appeared in Ref. 32 in the role of structure constants of certain inf-dim Lie algebra.

Looking at Eq. (30), it is easy to see that the generator $L^0_0 + U^0_0$ forms a center in $U(\infty + 1\,|\,\infty)$. $L^s_m\,(s \geq 1)$ form a basis in $su(\infty) \subset su(\infty + 1\,|\,\infty)$. L^1_m form a basis in su(2) and all the other generators are transformed under irreducible su(2)-representations.

The above considered superalgebras have a number of physical and geometrical realizations. So $su(\infty + 1\,|\,\infty)$ may be interpreted as an algebra of orthosymplectic superdiffeomorphisms on the coset supermanifold $SU(2\,|\,1)/U(1\,|\,1)$ (like $S^2 = SU(2)/U(1)$ for $su(\infty)$). Then $\underline{su}(\nu + 1\,|\,\nu)$ may be viewed as its quantum deformations (quantum operatorial algebras on this supermanifold serving as a phase space of certain quantum Bose-Fermi systems).

The other realizations of the bosonic algebra listed at the end of Sec. 2 may also be extended to the supercase. The realization (iii) for $\underline{sl}(\nu+1\,|\,\nu)$ can be constructed in terms of differential operators on the supercircle $S^{1|2}$. The realization (iv) of Ref. 5 describes directly both Bose and Fermi generators. Also it would be interesting to realize $\underline{sl}(\nu+1\,|\,\nu)$ as certain factor-algebra of the universal enveloping algebra of $sl(2\,|\,1)$, as in Refs. 1, 2, and 4 for $\underline{sl}(\nu)$.

5. Higher Spin Sugawara Construction and Related WZNW and Toda Models

Now with the families of inf-dim (super)algebras at our disposal, we are going to discuss extensions of various physical models involving classical Lie (super)algebras to the inf-dim case. Recently the large-N limits have become rather popular. At the beginning there is a discrete series of models based on the classical fin-dim algebras (su(N) for example). In the large-N limit one obtains a model based on inf-dim algebra (e.g., su(∞)). However, apart from integer N and $N = \infty$ it is also worthwhile to consider a whole continuous family of models, when N is an arbitrary real number. They may be based on the algebras considered in Refs. 1–5 and in the present paper. In this paper we will concentrate on a few examples of 2D QFT models.

Let g be some fin-dim simple Lie algebra. 2D quantum field theory naturally yields two closely related inf-dim generalizations of g. The first one is an affine Kac-Moody algebra $\hat{g}$, the chiral current algebra with the Schwinger term. The currents $J^a(z)$ can be realized for example through the fermionic fields, or bosonic fields in the WZNW model on the group manifold associated with g. The second generalization of g provided by 2D QFT is the so-called Casimir algebra associated with g. Let g have a range r and hence have r independent Casimir invariants. Then to any such invariant of order s one can associate a spin-s chiral current (see Refs. 13–17)

$$W^s(z) \sim : C^{(s)}_{a_1 \ldots a_s} \mathcal{J}^{a_1} \ldots \mathcal{J}^{a_s} : .$$

These r currents form a nonlinear algebra in the case of $s > 2$. In the simplest case of $g = \mathrm{sl}_2$, where only one independent Casimir invariant is available, one obtains the Virasoro algebra realized through the affine sl(2) algebra by means of the Sugawara construction. In the case of su_3 one obtains the so-called W_3 algebra[13] involving currents with spins 2 and 3. The characteristic feature of such algebras is their non-linearity and presence at only a finite number of higher spin currents.

Recently the problem of constructing linear Lie algebras involving, along with the Virasoro subalgebra, an infinite tower of higher spin generators like higher spin algebras in four-dimensional space-time[8,9,18–20] has attracted a lot of attention.[4,21,22,d]

Here we want to propose one of the ways to obtain generalizations of the Virasoro algebra with an infinite number of higher spin currents. It consists of replacing the fin-dim algebra g by its inf-dim generalization. So let us take, instead of the series A_N, continuously parametric series $\underline{A}_\nu$ ($0 \le \nu \le \infty$, we also have joint $\nu = \infty$, $\underline{A}_\infty \equiv \mathrm{su}(\infty)$ (for compact form)). We can introduce the corresponding affine Kac-Moody algebras $\hat{\underline{A}}_\nu$ for all non-exceptional points of ν including $\nu = \infty$. For the exceptional ones, $\nu = N$, $\underline{A}_N$ has the degenerate bilinear form (9), and to define Kac-Moody algebra it is necessary to pass to the factor-algebra, usual fin-dim A_N, to make it invertible. Having at our disposal the affine algebra $\hat{\underline{A}}_\nu$ we can consider a generalized higher spin Sugawara construction. A crucial feature is that, owing to the presence of infinitely many generators at the general position points of ν (including ∞), A_ν has an infinite number of independent Casimir invariants and we are able to introduce an infinite number of higher spin currents

$$W^s(z) \sim : \mathrm{tr}(\underbrace{\mathcal{J}(z) \ldots \mathcal{J}(z)}_{s}) : , \quad s = 2, 3, \ldots ,$$

$$\mathcal{J}(z) = \Sigma \mathcal{J}^l_m(z) T^l_m . \tag{32}$$

[d] It should be mentioned that, to our knowledge, higher spin (super)algebras involving infinitely many higher spin generators firstly were discovered in Refs. 8 and 9 in the context of constructing the gauge theory for massless higher spin fields in the anti-de Sitter universe.[23] The conformal higher spin superalgebras with an infinite tower of higher spins have been investigated in Refs. 18–20, 4 and 25 in various dimensions of space-time.

However, the price for infinite dimension in this construction consists of divergencies of the central charges and the prefactors. Thus the Sugawara energy-momentum for $\underline{su}(\nu)$ and $su(\infty)$ has a divergent central charge because it includes the factor dim g. Nevertheless the finite answer may be obtained by using some appropriate regularization (e.g. ζ-function one).

However a cardinal way out of the situation consists of introducing the supersymmetry. To do it let us substitute $\underline{su}(\nu+1\,|\,\nu)$ and $su(\infty+1\,|\,\infty)$ for $\underline{su}(\nu)$ and $su(\infty)$. Although these superalgebras are inf-dim, the characteristics we worry about are the finite ones. So their graded dimensions are equal to zero:

$$s\dim \underline{su}(\nu+1\,|\,\nu) = s\dim su(\infty+1\,|\,\infty)$$

$$= \sum_{s=1}^{\infty}\Big(\sum_{m=-s}^{\infty} 1 - 2\sum_{m=-s+1/2}^{s-1/2} 1 + \sum_{m=-s+1}^{s-1} 1\Big) = 0\,. \tag{33}$$

The Virasoro central charge and the prefactor in $T(z) = 1/2\beta : \mathrm{tr}(L^2)$: for the affine Sugawara construction for some Lie superalgebra G are given by[15]:

$$c = \frac{k\, s\dim G}{k+g}\,, \quad \beta = k+g\,, \tag{34}$$

similar to the bosonic case. For the superalgebras in question we have

$$c = 0 \quad \text{and} \quad \beta = 1+k\,. \tag{35}$$

The secret of finiteness for the inf-dim superalgebras at issue is accounted for by the known fact that invariants of superalgebras $SU(N\,|\,M)$ depend on $N-M$ as for the ordinary algebras $SU(N-M)$.[25,15] The non-zero central charge will appear if one adds the u(1) factor and considers $u(\infty+1\,|\,\infty)$, $\underline{u}(\nu+1\,|\,\nu)$. In this way, working with the superalgebras one is able, in principle, to calculate all the OPE's of the higher spin currents, i.e., construct manifestly the corresponding Casimir superalgebras with an infinite tower of currents.

A crucial problem is whether they are linear Lie algebras or nonlinear ones. At the very beginning it is not easy to verify that these currents can form a usual linear algebra. However, recently an example of linear algebra with central terms for all spins was constructed in Ref. 22. It might be that the algebra W_∞ as a matter of fact, appears naturally as a Casimir algebra via the higher spin Sugawara construction for $su(\infty)$. Note that the above discussion leads to a whole continuous family of the higher spin generalization of the Virasoro algebra. However, the problem as to whether they are all linear, or, e.g., only one corresponds to $su(\infty)$, is still open.

Now let us discuss the concrete models realizing $\underline{SU}(\nu)$, $SU(\infty)$ and their Casimir algebras. The affine algebra $\hat{g}$ for fin-dim g naturally appears in the WZNW models[26] on the group manifold. Similarly one might consider at the very beginning the WZNW models based on the inf-dim group manifolds (e.g., $\underline{SU}(\nu)$, $SU(\infty)$ = SDIIF(S^2)). The corresponding WZNW action has a usual form, where $g^{-1}\partial_\mu g$ belongs to the inf-dim algebra and tr is the trace operation in this algebra discussed above. At the exceptional points $\nu = N$ such an action becomes the usual WZNW action for $SU(N)$ due to the degeneracy of the bilinear form $\mathrm{tr}(AB)$. When $g \in$

SDIIF(S^2) and $g^{-1}\partial_\mu g \in$ su(∞), the tr operation is, in fact, an integration over S^2 and the WZNW action looks like an effective four-dimensional model. Similar construction for Chern-Simons theory was considered in Refs. 3 and 5.

As was shown in Ref. 17, Toda model associated with some fin-dim algebra g can be obtained from the corresponding WZNW model by imposing certain special constraints on the currents. Following this method in the case of inf-dim algebras $\underline{A}_\nu$, one might obtain a continuously parametric family of Toda models. To do it, one should pass to a new Cartan-Weyl basis for $\underline{A}_\nu$. The generators T_0^s, $s = 1, 2, \ldots$ form a basis in a maximal (inf-dim) commutative subalgebra in $\underline{A}_\nu$. Now they should be diagonalized simultaneously. If it is indeed possible for any non-exceptional ν and $\nu \to \infty$, (for exceptional $\nu = N$ we replace the non-simple $\underline{A}_N$ for its simple factor-algebra A_N) one would obtain an infinite set of roots and, in particular, simple roots α_i, $i = 1, 2, \ldots$. The Cartan matrix would be an infinite matrix $\mathcal{K}_{ij}(\nu) = \mathcal{K}_{\alpha_i\alpha_j}(\nu)$ with components depending on ν. Then the corresponding Toda model would involve an infinite set of fields Φ_i obeying the Toda equations of motion with $\mathcal{K}_{ij}(\nu)$ in place of the Cartan matrix for fin-dim simple algebra.

The Casimir algebra derived from the WZNW model currents, when the constraints of Ref. 17 are taken into account by means of the Dirac brackets, is transformed into the W-algebra of Toda model.[16] Naturally the notation $\underline{W}_\nu$ may be introduced for the models based on $\underline{A}_\nu$ (and W_∞ for A_∞(su(∞)).

It should be mentioned that in Ref. 27 certain continuous Toda models were considered based upon the continuous Lie algebras with a certain operator $\mathcal{K}(\tau, \tau')$ (for example $\delta''(\tau - \tau')$) in place of the Cartan matrix. It is interesting to establish the correspondence between the algebras $\underline{\mathrm{sl}}(\nu)$ and the continuous algebras of Ref. 27. For example, the Cartan matrix of quantized diffeomorphisms algebra might be represented as a differential operator $\mathcal{K}(\tau, \tau' \mid \nu) = \mathcal{K}_\nu(\partial/\partial\tau)\, \delta(\tau - \tau')$, and the corresponding Toda model might be viewed as some effective three-dimensional model.

Another way to construct $\underline{W}_\nu$ and W_∞ algebras may consist of using the quantum Hamiltonian reduction method for $\underline{A}_\nu$ and A_∞, as was done for fin-dim algebras A_N in Ref. 28.

Also, starting with the superalgebras $\underline{\mathrm{sl}}(\nu + 1 \mid \nu)$ and sl($\infty + 1 \mid \infty$), as we have discussed above, the superalgebras $\underline{W}_{\nu+1|\nu}$ and $W_{\infty+1|\infty}$ might be considered manifestly.

6. Conclusion

Among the applications, without doubt the most interesting, and problematic at the same time, is whether certain hypothetical string theories with an infinite tower of higher spin gauge fields living on the worldsheet may exist. In this connection an infinite tower of higher spin gauge fields in $D = 1 + 1$ was considered in Ref. 20 and the conformal anomalies for the ghost sector were calculated in Ref. 4. In Ref. 29 a finite and in Ref. 30 an infinite set of higher spin gauge fields interacting with the scalar fields have been considered respectively.

However, at present the problem as to whether strings with worldsheet higher spin symmetry may exist at all is still open. To speculate about their properties if the answer is positive, it would be of interest to examine their massless sector. It might be especially esthetical if the massless sector contains an infinite tower of massless higher spin gauge fields of all spins in the space-time. However, according to the results of Ref. 23, such (interacting) strings with unbroken higher spin gauge symmetry might propagate only on the anti-de Sitter background with non-zero cosmological constant. On the other hand, if certain connections between the worldsheet and space-time higher spin gauge symmetries may really take place, all the attempts to construct interacting string theory with higher spin worldsheet gauge symmetry will not be a success without introducing a non-trivial anti-de Sitter background.[e]

Another problem of considerable interest consists of developing an algebraic theory in $D > 2$ space-time analogous to the theory of the Virasoro algebra and its higher spin generalizations in $D = 2$. Formally, from the algebraic point of view, the algebra sl_2 is not singled out. In Refs. 31 and 32 we have developed a formal construction (analytic continuation of the semisimple fin-dim Lie algebras) which puts certain Virasoro-like inf-dim algebra in correspondence with a fin-dim semisimple algebra g. It turned out that there also exist higher spin generalizations of those algebras like W_∞ for the Virasoro one. In particular in Ref. 32 the Virasoro-like generalization of so(3,2) (conformal algebra in $D = 2 + 1$) and its higher spin extension have been considered in detail.

To conclude, we would like to mention that the Racah basis for $SU(N+1\,|\,N)$ constructed in this paper may be of use in atomic and nuclear physics, similar to the Racah basis for $SU(N)$.[11]

Appendix

For any three integer or half-integer numbers (s, s', s'') satisfying the triangle condition $|s-s'| \leq s'' \leq s+s'$, and for all $k = -s, -s+1, \ldots, s$; $k' = -s', -s'+1, \ldots, s'$; $k'' = -s'', -s''+1, \ldots, s''$ we define a symbol

$$f^{ss's''}_{kk'k''}(v) = \sqrt{2s''+1}\,\Delta(s,s',s'')\,\delta(k+k'-k'')$$
$$\times \sum_t (-1)^t \{ \prod_{p=1}^{s+s'-s''-t} (v-s''+k''-p) \prod_{q=1}^{t} (v+s''+k''+q) \}$$
$$\times \frac{\sqrt{(s+k)!\,(s-k)!\,(s'+k')!\,(s'-k')!\,(s''+k'')!\,(s''-k'')!}}{t!(s+s'-s''-t)!\,(t+s''-s-k')!\,(t+s''-s'+k)!\,(s-k-t)!\,(s'+k'-t)!} \tag{A1}$$

[e] This strong statement is based on the known negative result ("no-go" theorem) that there does not exist any non-trivial gauge-invariant interaction among higher spin ($s > 2$) gauge fields and Einstein gravity without cosmological term. On the other hand, the positive results of Ref. 23 assert that there does exist a non-trivial interaction on the anti-de Sitter background, but it is non-analytical on the cosmological constant. Another variant discovered in Ref. 24 consists of the existence of conformally-invariant interaction of higher spin gauge fields with Weyl gravity (at least in the cubic order). All these arguments should be taken into account when the possibilities of constructing asymptotical symmetric phases of string theory are considered (see the second reference in Ref. 24).

$$\Delta(s,s',s'')=\left[\frac{(s+s'-s'')!\,(s-s'+s'')!\,(s'+s''-s)!}{(s+s'+s''+1)!}\right]^{1/2}. \tag{A2}$$

So the defined symbol $f(v)$ is a polynomial function of the real parameter, $v\in\mathbb{R}$. Due to the simple property

$$f^{ss's''}_{kk'k''}(-v)=f^{ss's''}_{-k,-k',k''}(v), \tag{A3}$$

we can set $v\geq 0$ without losing of information. When $v=N=1,2,\ldots$ is integer and

$$s\leq N-1,\; s'\leq N-1,\; s''\leq N-1, \tag{A4}$$

our symbol is directly reduced to the $6j$-symbol as follows,

$$\begin{aligned}f^{ss's''}_{kk'k''}(N)&=\delta(k+k'-k'')\sqrt{2s''+1}\,(-1)^{s''+k''+N-1}\\&\times\left[\frac{(N+s-k'+k'')!\,(N+k'+s')!\,(N+k''-s''-1)!}{(N-k'+k''-s-1)!\,(N+k'-s'-1)!\,(N+k''+s'')!}\right]^{1/2}\\&\times\left\{\begin{matrix} s & s' & s''\\ \frac{N-1}{2}, & \frac{N-1}{2}+k'', & \frac{N-1}{2}+k'\end{matrix}\right\}.\end{aligned} \tag{A5}$$

In this way the symbol (A1) in fact is an analytic continuation of the $6j$-symbol when N becomes arbitrary real and the restrictions (A4) are abolished.

An easily verified peculiar feature of f consists of its large-v behavior at $v\to\infty$,

$$\lim_{v\to\infty}[v^{-s-s'+s''}f^{ss's''}_{kk'k''}(v)]=C^{ss's''}_{kk'k''}, \tag{A6}$$

$$\begin{aligned}f^{ss's''}_{kk'k''}(v)&=v^{s+s'-s''}C^{ss's''}_{kk'k''}\\&\quad-v^{s+s'-s''-1}\Big\{\sqrt{(s-k)(s'+k')(s+s'-s'')(s+s'+s''+1)}\\&\quad\times C^{s-1/2,\,s'-1/2,\,s''}_{k+1/2,\,k'-1/2,\,k''}+\frac{1}{2}(s+s'-s'')(s+s'+s''-2k''+1)C^{ss's''}_{kk'k''}\Big\}\\&\quad+\text{lower powers of } v.\end{aligned} \tag{A7}$$

This large-v behavior plays a central role in the calculation of large-N limits for the various Lie algebras and superalgebras in the $su_2(sl_2)$-irreducible basis.

In view of (A6) and (A7), $f(v)$ may be interpreted as a quantum deformation of the Clebsh-Gordan coefficients with the deformation parameter $\hbar=1/v$. They are restored in the classical limit $\hbar\to 0$. This phenomenon is well-known for the $6j$-symbols. In particular our symbols $f^{ss's''}_{kk'k''}(v)$ are proportional to the symbols $W^{ss's''}_{kk'k''}(j)$ considered in Ref. 11 (see Eqs. (3.278) and (3.300)) at the integer points $v=N$ and $j=(N-1)/2$. However, $f(v)$ is defined for all real values of v and the restrictions (A4) are not required.

For the reader's convenience, we also present here the interesting formula,

$$\sum_{M_1, M_2, M_3} C^{\jmath_2 j_1 \jmath_3}_{M_2 m_1 M_3} C^{\jmath_3 j_2 \jmath_1}_{M_3 m_2 M_1} C^{\jmath_1 j_3 \jmath_2}_{M_1 -m_3 M_2}$$
$$= (-1)^{j_1+j_2+2\jmath_3+m_3} \left[\frac{(2\jmath_1+1)(2\jmath_2+1)(2\jmath_3+1)}{(2j_3+1)}\right]^{1/2}$$
$$\times \begin{Bmatrix} j_1 & j_2 & j_3 \\ \jmath_1 & \jmath_2 & \jmath_3 \end{Bmatrix} C^{j_1 j_2 j_3}_{m_1 m_2 m_3} . \tag{A8}$$

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