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# Conformal Superalgebras of Higher Spins

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Infinite-dimensional conformal higher-spin superalgebras in four-dimensional space-time are constructed. The superalgebra of conformal supergravity is a maximal finite-dimensional subalgebra of our higher-spin conformal superalgebras. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

In the papers [1, 2] we developed the superconformal theory of higher spins in three-dimensional space-time. Our construction was based on the infinite-dimensional superalgebra shsc($N|3$), which generalizes the usual conformal superalgebra osp($N|4$) in the space-time with $D=2+1$. The aim of this work is to construct the infinite-dimensional generalizations of conformal superalgebras SU($2,2|N$) in $D=3+1$ [4]. The gauge theories corresponding to these superalgebras describe all spins from 1 to infinity and include, in particular, the usual conformal supergravity. As it was mentioned in Refs. [1–3], these theories can give a consistent description of interaction of the gauge fields of higher spins with the Weyl gravity.

Our construction is based on the method of operator realizations of Lie superalgebras [5, 6] and on the theory of symbols of operators [6, 8].

Using the method of operator realizations we construct a series of infinite-dimensional complex superalgebras igl($M|N; \mathbb{C}$) (*i*-infinite-dimensional), generalizing a series of superalgebras gl($M|N; \mathbb{C}$). Factorising these superalgebras with respect to their centre ($M\neq N$), we obtain a series of superalgebras isl$^\infty$($M|N; \mathbb{C}$), generalizing the series sl($M|N; \mathbb{C}$). These superalgebras contain (with infinite degeneracy) all irreducible representations sl($M|N; \mathbb{C}$) with signatures, proportional to the signature of the adjoint representation sl($M|N; \mathbb{C}$). The factoralgebras isl($M|N; \mathbb{C}$) of algebras isl$^\infty$($M|N; \mathbb{C}$) with respect to their radical contain all such representations only once and are simple. They directly generalize the series of simple ($N\neq M$) superalgebras sl($M|N; \mathbb{C}$). These results, combined with the results of Refs. [6, 12] are represented in Table I.

The superalgebras shs′ and isl have a common structure. They are simple and



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TABLE I

| Finite-dimensional complex superalgebras | Infinite-dimensional generalizations | Superalgebra |
|---|---|---|
| $osp(N\|M;\mathbb{C})$ | $shs'^{E}(N\|M;\mathbb{C})$ | Simple |
| $gl(M\|N;\mathbb{C})$ | $igl(M\|N;\mathbb{C})$ | Not simple |
| $sl(M\|N;\mathbb{C})$ | $isl^{\infty}(M\|N;\mathbb{C})$<br>$isl^{(n)}(M\|N;\mathbb{C})$<br>$isl(M\|N;\mathbb{C})$ | Not simple<br>Not simple, $n=2, 3, \ldots$<br>simple, when $M \neq N$ |

contain all irreducible representations, with signature proportional to the signature of the adjoint representation of maximal finite-dimensional subalgebra osp or sl, only once. It is also interesting to find similar infinite-dimensional Lie algebras and superalgebras for all Cartan series of Lie algebras and superalgebras and for exceptional types of Lie superalgebras and to obtain their complete classification.

The corresponding real forms of superalgebras $isl^{\infty}(4|N;\mathbb{C})$ and $isl(4|N;\mathbb{C})$ generalize the conformal superalgebras $SU(2,2|N)$.

However, in order to construct gauge theories, it is necessary to introduce in these superalgebras a special basis, in which all the generators have definite Lorenzian $sl(2;\mathbb{C})$ structure, $su(N)$-structure and fixed conformal and chiral weights (in this basis the operators of chirality and dilatations are diagonal). We present a detailed construction of such a basis (we call it superconformal) for $N=1$ superalgebras $isu^{\infty}(2,2|1)$ and $isu(2,2|1)$. In the following we denote the algebras $isu^{\infty}(2,2|1)$ and $isu(2,2|1)$ as $shsc^{\infty}(4|1)$ and $shsc(4|1)$ (super higher-spin conformal).

The structure coefficients of superalgebras $shsc^{(\infty)}(4|1)$ are calculated by us in the superconformal basis. The gauge field, corresponding to the $shsc^{\infty}(4|1)$, contains an infinite number higher-spin conformal supermultiplets $(s, s-\frac{1}{2}, s-1)$ with highest spin $s=2, 3, \ldots, \infty$. The gauge field of $shsc(4|1)$ (unlike the case of $shsc^{\infty}(4|1)$) contains only one conformal supermultiplet for every higher spin. The curvatures of $shsc^{(\infty)}(4|1)$ generalize the curvatures of usual conformal supergravity [4, 9, 10] to the case of all higher spins.

The results of this paper and of Refs. [1–3, 6, 11–14] are illustrated in the Table II.

The global algebra and its localization in the theory of higher spins in anti-de Sitter space-time were obtained in [6, 11–14].

In the papers [1, 2] the superconformal algebra of higher spins and its localization were constructed in $D=2+1$. In Ref. [3] we briefly consider the conformal theory of higher spins in $D=3+1$. The superconformal theory of higher spins in

TABLE II

| Algebras of supergravity theories | Infinite-dimensional generalizations of supergravity algebras and corresponding gauge theories |
|---|---|
| $adS_4$-supergravity<br>$osp(N\|4)$ | $adS_4$ higher-spin theory<br>$shs^E(N\|4)$ |
| $D=4$ conformal supergravity<br>$SU(2,2\|N)$ | Conformal theories of higher spins in $D=4$,<br>$shsc^\infty(4\|N)$, $shsc(4\|N)$ |
| $adS_3$-supergravity<br>$osp(M\|2)\oplus osp(N\|2)$ | Theory of higher spins in $adS_3$,<br>$shs^E(N\|2)\oplus shs^E(M\|2)$ |
| $D=3$ conformal supergravity<br>$osp(N\|4)$ | Conformal theories of higher spin in $D=3$,<br>$shsc(N\|3)$ |

$D=3+1$ based on the global superconformal algebra, proposed in the present paper, will be considered by us in detail in another paper.

Such superconformal theories are interesting due to a number of reasons:

First, as was already mentioned in Ref. [1], it is possible that one could construct a consistent interaction of higher spins among themselves and with Weyl gravity in all orders in interaction due to the expected closure of the gauge algebra (in analogy to conformal supergravity). The extended superconformal theories of all spins would be particulary interesting (see Section 8), if it would be possible to realize a variant of spontaneous breaking of conformal symmetry, leading to Einstein gravity, interacting with massive higher spins, and therefore to reveal a connection with the string theory.

Another possible variant of the spontaneous breaking of conformal symmetry can lead to the generation of a cosmological constant, with the transformation of the conformal theory of higher spins into $adS_4$-theory. In this case in the corresponding $adS_4$-theory there arises a full compleat of auxiliary fields, that are necessary for the closure of gauge algebra and which permit progress in the construction of the full interaction of theories [11–14] and in the proof of its invariance.

To conclude, let us discuss the strategy for constructing higher spin conformal superalgebras. Recall first the strategy for constructing the $adS_4$ superalgebra $shs^E(N|4)$ which appeared in [6].

The first step is to choose a convenient operator realization of the finite-dimensional subalgebra $osp(N|4)$. To do that, we choose the generating elements

$$q_\alpha^+ = r_{\dot\alpha}, \qquad [q_\alpha, q_\beta] = 2i\varepsilon_{\alpha\beta}, \qquad [r_{\dot\alpha}, r_{\dot\beta}] = 2i\varepsilon_{\dot\alpha\dot\beta}, \tag{1.1}$$

$$\psi_i^+ = \psi_i, \qquad \{\psi_i, \psi_j\} = 2\delta_{ij}, \qquad i, j = 1, ..., N, \tag{1.2}$$

with $\alpha$ and $\dot{\alpha}$ as two-component spinor indices. Second-order polynomials[1] in these generating elements furnish the osp$(N|4)$ Lie superalgebra with respect to the commutator,

$$[A, B\} = A \cdot B - (-1)^{\varepsilon(A)\,\varepsilon(B)} B \cdot A, \tag{1.3}$$

where the Grassmann parity of quadratic operators is defined by

$$A(-q, -r, \psi) = (-1)^{\varepsilon(A)} A(q, r, \psi) \tag{1.4}$$

and the dot denotes the associative operator multiplication.

The next step is to consider polynomials of arbitrary even degree in the generating elements. They furnish the associative algebra aq$^E(N|4; \mathbb{C})$ (associative quantum). Introducing in aq$^E(N|4; \mathbb{C})$ a Lie superalgebra structure by the relation (1.3), and extracting the real form we get an infinite-dimensional superalgebra shs$^E(N|4)$ [6]. The generators of shs$^E(N|4)$ carry a higher spin adS$_4$-superalgebra osp$(N|4)$ representation, while the corresponding gauge fields describe higher spins in adS$_4$. Explicit expressions for the shs$^E(N|4)$ curvatures are easily obtained within the theory of symbols of operators, using convenient formulae for the multiplication of Weyl symbols.

In the present work an analogous method is used to construct a higher-spin extension of the SU$(2, 2|N)$ conformal superalgebra. Similarly to the adS$_4$-case, we start with an operator realization of SU$(2, 2|N)$. Note that the su and osp superalgebras are of different natures and thus have different operator realizations. The osp superalgebra is furnished by all polynomials which are quadratic in the Heisenberg operators, whereas the conformal superalgebra is furnished by those operators which commute with the "particle number" operator. To realize the SU$(2, 2|N)$ algebra, a convenient choice of generating elements is

$$[a^\alpha, \bar{a}_\beta] = 2\delta^\alpha_\beta, \qquad [a_{\dot{\alpha}}, \bar{a}^{\dot{\beta}}] = 2\delta^{\dot{\beta}}_{\dot{\alpha}},$$

$$\{\alpha_i, \alpha_j^+\} = 2\delta_{ij}, \qquad i, j = 1, \ldots, N, \tag{1.5}$$

$$(a^\alpha)^+ = \bar{a}^{\dot{\alpha}}, \qquad (\bar{a}_\beta)^+ = a_{\dot{\beta}}, \qquad (\alpha)^+ = \alpha^+. \tag{1.6}$$

All operators quadratic in $(a, \bar{a}, \alpha, \alpha^+)$ furnish osp$(2N|8)$, whereas those commuting with the "particle number" operator

$$T = \bar{a}_\alpha a^\alpha + \bar{a}^{\dot{\beta}} a_{\dot{\beta}} + \alpha^+ \alpha \tag{1.7}$$

(excluding the operator (1.7) itself) form a subalgebra in osp$(2N|8)$ which is isomorphic to SU$(2, 2|N)$.

Our next step is to consider of higher order polynomials in the generating elements. The associative algebra of those polynomials in the generating elements

[1] All operators have the Weyl (symmetric) ordering.

(1.5), (1.6) that commute with the "particle number" operator is called aqpc(4 | $N$; $\mathbb{C}$) (associative quantum particle conservation). The Grassmann parity in aqpc(4 | $N$; $\mathbb{C}$) is defined as

$$A(-a, -\bar{a}, \alpha, \alpha^+) = (-1)^{\varepsilon(A)} A(a, \bar{a}, \alpha, \alpha^+). \tag{1.8}$$

The real Lie superalgebra structure in the associative algebra aqpc(4 | $N$; $\mathbb{C}$) is introduced with the help of the commutator (1.3) and antihermiticity condition $A^+ = -A$ with Hermitian conjugation (1.6). We denote this superalgebra as iu(2, 2 | $N$) (infinite-dimensional unitary).

To construct a gauge theory, however, it is necessary to introduce in iu(2, 2 | $N$) a particular superconformal basis, in which iu(2, 2 | $N$) would be explicitly decomposed into SU(2, 2 | $N$) irreducible representations.

To construct this basis, the standard technique of the representation theory of Lie algebras is to be used. One finds all the highest vectors with maximal conformal weight and acts on them with operators which lower conformal weight. Then irreducible spinor basis is obtained through the use of spinorial Clebsch–Gordan sl(2; $\mathbb{C}$) coefficients. The curvatures are calculated in the superconformal basis with the help of the known structure coefficients for the three-dimensional conformal superalgebra shsc($N$ | 3).

Factorising the iu(2, 2 | $N$) with respect to its centre, generated by powers of the "particle number" operator (1.7), one obtains the superalgebra shsc$^\infty$(4 | $N$). The shsc$^\infty$(4 | $N$) superalgebra contains a family of ideals, which are embedded into each other. Factoring out these ideals, we obtain a family of factoralgebras shsc$^{(n)}$(4 | $N$), $n = 1, 2, ..., \infty$. They contain conformal supermultiplets with multiplicity equal to $n$. The superalgebra shsc$^{(1)}$(4 | $N$), denoted simply as shsc(4 | $N$), is simple (for $N \neq 4$) and contains all conformal supermultiplets only once.

All the above superalgebras are candidates for the role of a physical conformal higher-spin symmetry.

The next problem is the localization of the proposed superalgebras and the construction of a complete Lagrangian. It is only at this stage that one will single out a higher-spin conformal superalgebra which will allow one to construct a gauge-invariant interacting higher-spin theory.

This paper is organized as follows: In Section 2 we construct the complex superalgebras igl($M$ | $N$; $\mathbb{C}$), isl$^\infty$($M$ | $N$; $\mathbb{C}$), and isl($M$ | $N$; $\mathbb{C}$) and analyze their structure. In Section 3 we introduce a convinient for our purposes operator realization of SU(2, 2 | 1). In Section 4 we consider the representations of conformal algebra SU(2, 2) with higher spins (Bose-case) and introduce a conformal basis in the representation spaces of SU(2, 2). In Section 5, using the results of Section 4, we construct the algebras hsc$^\infty$(4) and hsc(4) and discuss their properties. In Section 6 we generalize the results of Section 4 on the supercase and introduce a superconformal basis in the representation spaces of SU(2, 2 | 1). In Section 7 we construct the superalgebras shsc$^\infty$(4 | 1) and shsc(4 | 1), which generalize in the supercase the algebras hsc$^{(\infty)}$(4), constructed in Section 5, and study their structure. In Section 8 we briefly discuss the extended superalgebras $\mathrm{shsc}^{\infty}(4|N)$ and $\mathrm{shsc}(4|N)$ and outline the possible perspectives of their application. In Section 9 the higher-spin conformal superalgebras $\mathrm{shsc}^{(n)}_{\rho}(4|N)$ are considered.

At the end of the paper we place several appendices, in which we describe some calculations, and we also give a brief description of the results of Refs. [1, 2] that are essential for the construction of the conformal superalgebra of higher spins in $D=3+1$.

## 2. Infinite-Dimensional Complex Lie Superalgebras $\mathrm{igl}(M|N;\mathbb{C})$, $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$, and $\mathrm{isl}(M|N;\mathbb{C})$

Let us consider the Heisenberg–Clifford superalgebra

$$[\hat{a}_A, \hat{a}_B\} = 2\mathbb{C}_{A,B}, \qquad A, B = 1, ..., N+2M, \tag{2.1}$$

where $\hat{a}_A$ are the generating elements with grassmanian parity $\varepsilon(\hat{a}_A) = \varepsilon_A$ and nondegenerate orthosymplectic metric

$$\mathbb{C}_{A,B} = -(-1)^{\varepsilon_A \varepsilon_B}\, \mathbb{C}_{B,A}, \qquad (1-(-1)^{\varepsilon_A+\varepsilon_B})\, \mathbb{C}_{A,B} = 0, \tag{2.2}$$

where $2M$ and $N$ is correspondingly the number of even and odd generating elements. The nondegenerate orthosymplectic metric can be chosen in the form

$$\mathbb{C}_{A,B} = \begin{array}{c} N \\ 2M \end{array}\left(\begin{array}{c|c} I & 0 \\ \hline 0 & \begin{matrix} i\varepsilon & & \\ & \ddots & \\ & & i\varepsilon \end{matrix} \end{array}\right), \qquad \begin{array}{cc} N & 2M \end{array} \tag{2.3}$$

where

$$i\varepsilon = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{2.4}$$

$$I = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{N} N, \quad I\text{-unit matrix.} \tag{2.5}$$

We denote by $a_A$ the symbols of operators $\hat{a}_A$ with $\varepsilon(a_A) = \varepsilon(\hat{a}_A) = \varepsilon_A$ and consider the associative algebra of Weyl symbols of the operators, *polynomial* with respect to $a_A$, aq$(N \mid 2M; \mathbb{C})$ (associative quantum) [6]. The associative product of the Weyl symbols is given by the formula

$$A * B = A \exp(\overleftrightarrow{\Delta}) B, \tag{2.6}$$

where $\overleftrightarrow{\Delta}$ is a differential operator, acting simultaneously to the left and to the right, and

$$\overleftrightarrow{\Delta} = \frac{\overleftarrow{\partial}}{\partial a_A} \mathbb{C}_{A,B} \frac{\overrightarrow{\partial}}{\partial a_B}. \tag{2.7}$$

The natural basis in aq$(N|2M; \mathbb{C})$ is given by the monomials

$$T_{A_1 \cdots A_n} = \frac{1}{n!} a_{A_1} \cdots a_{A_n}, \qquad n = 0, 1, 2, .... \tag{2.8}$$

The basis monomials (2.8) are symmetric:

$$T_{A_1 \cdots A_n} = \mathrm{Sym}(T_{A_1 \cdots A_n}). \tag{2.9}$$

The generalized symmetrization is defined with the help of symmetrizators of the form [8]:

$$n! \, S^{A'_1 \cdots A'_n}_{A_1 \cdots A_n} = \left( \frac{\overrightarrow{\partial}}{\partial a_{A'_1}} \cdots \frac{\overrightarrow{\partial}}{\partial a_{A'_n}} a_{A_n} \cdots a_{A_1} \right) \tag{2.10}$$

by the formula

$$\mathrm{Sym}(T_{A_1 \cdots A_n}) = S^{A'_1 \cdots A'_n}_{A_1 \cdots A_n} T_{A'_1 \cdots A'_n}. \tag{2.11}$$

In the following the symmetrizations in the expressions of the type of $\mathrm{Sym}(X_{A_1 \cdots A_n, B_1 \cdots B_m, \ldots})$ is performed separately with respect to all groups of indices denoted by the same letters.

The multiplication law (2.6) in this basis takes the form

$$T_{A_1 \cdots A_n} * T_{B_1 \cdots B_m} = \sum_k \frac{(n+m-2k)!}{k! \, (n-k)! \, (m-k)!} \times \mathrm{Sym}(\mathbb{C}_{A_n, B_1} \cdots \mathbb{C}_{A_{n-k+1}, B_k} T_{A_1 \cdots A_{n-k} B_{k+1} \cdots B_m}). \tag{2.12}$$

In the following we shall consider an algebra aq$^E(N|2M; \mathbb{C})$ generated by the elements $A \in$ aq$(N|2M; \mathbb{C})$ such that

$$A(a_A) = A(-a_A). \tag{2.13}$$

Let us define in the algebra $\mathrm{aq}^E(N|2M;\mathbb{C})$ the structure of Lie superalgebra by the relation[2]

$$[A, B\} = A * B - (-1)^{\varepsilon(A)\varepsilon(B)} B * A, \tag{2.14}$$

where the grassmanian parity of the basis monomials is given by the formula

$$\varepsilon(T_{A_1 \cdots A_n}) = \varepsilon_{A_1 \cdots A_n} = \sum_{i=1}^{n} \varepsilon_{A_i}. \tag{2.15}$$

Then the commutation relations in this algebra take the form

$$[T_{A_1 \cdots A_n}, T_{B_1 \cdots B_m}\} = 2 \sum_k \frac{(n+m-2k)!}{k!\,(n-k)!\,(m-k)!}\,\delta(|k+1|_2) \times \mathrm{Sym}(\mathbb{C}_{A_n, B_1} \cdots \mathbb{C}_{A_{n-k+1}, B_k} T_{A_1 \cdots A_{n-k} B_{k+1} \cdots B_m}). \tag{2.16}$$

This superalgebra coincides with the superalgebra $\mathrm{shs}^E(N|2M;\mathbb{C})$, first obtained in Ref. [6]. The quadratic elements with the basis $T_{A_1A_2}$ in $\mathrm{shs}^E(N|2M;\mathbb{C})$ form the maximal finite-dimensional subalgebra, isomorphic to $\mathrm{osp}(N|2M;\mathbb{C})$ with the commutator[3]

$$[T_{A_1A_2}, T_{B_1B_2}\} = 4\,\mathrm{Sym}(\mathbb{C}_{A_2B_1} T_{A_1B_2}). \tag{2.17}$$

The series of infinite-dimentional superalgebras $\mathrm{shs}^E(N|2M;\mathbb{C})$ generalizes the series $\mathrm{osp}(N|2M;\mathbb{C})$.

Let us consider the special choice of the commutation relations (2.1),

$$[\hat{a}_A, \hat{\bar{a}}^B\} = 2\delta_A^B, \qquad \varepsilon(\hat{a}_A) = \varepsilon(\hat{\bar{a}}^A) = \varepsilon_A, \qquad A, B = 1, \ldots, N+M. \tag{2.18}$$

The associative product of symbols has the form (2.6), where

$$\vec{\Delta} = \frac{\overleftarrow{\partial}}{\partial a} \cdot \frac{\vec{\partial}}{\partial \bar{a}} - \frac{\vec{\partial}}{\partial a} \cdot \frac{\overleftarrow{\partial}}{\partial \bar{a}}, \qquad \bar{a} \cdot a = \bar{a}^A a_A. \tag{2.19}$$

Let us consider the set of elements $A \in \mathrm{aq}^E(2N|2M;\mathbb{C})$ such that[4]

$$[T, A] = 0, \qquad T = \bar{a} \cdot a, \tag{2.20}$$

[2] The commutators of the Weyl symbols is defined as in (2.14) everywhere.

[3] Let us write (2.17), for example, for the case of $T_{i_1 i_2} = \frac{1}{2} a_{i_1} a_{i_2}$, $\varepsilon_i = 1$, $\{a_i, a_j\} = 2\delta_{ij}$. From our definitions it follows that

$$[T_{i_1 i_2}, T_{j_1 j_2}] = \delta_{i_1 j_2} T_{i_2 j_1} - \delta_{i_1 j_1} T_{i_2 j_2} + \delta_{i_2 j_1} T_{i_1 j_2} - \delta_{i_2 j_2} T_{i_1 j_1},$$

i.e., the usual commutation relations of $SO(N)$.

[4] For the definition of the commutators of symbols see (2.14), (2.6).

where $T$ is a symbol of the "particle number" operator. This set evidently forms a subalgebra in $\mathrm{aq}^E(2N|2M;\mathbb{C})$. Let us denote it by $\mathrm{aqpc}(N|M;\mathbb{C})$ (pc = particle conservation). It is natural to choose the basis in this algebra in the form

$$T^{A_1\cdots A_n}{}_{,B_1\cdots B_n}=\frac{1}{2(n!)^2}\,\bar{a}^{A_1}\cdots\bar{a}^{A_n}a_{B_1}\cdots a_{B_n}. \tag{2.21}$$

The grassmanian parity of monomials (2.21) is defined by the relation

$$\varepsilon(T^{A_1\cdots A_n}{}_{,B_1\cdots B_n})=\varepsilon^{A_1\cdots A_n}_{B_1\cdots B_n}=\sum_{i=1}^{n}(\varepsilon^{A_i}+\varepsilon_{B_i}). \tag{2.22}$$

With respect to the commutator (2.14) aqpc forms a Lie superalgebra, which we denote by $\mathrm{igl}(M|N;\mathbb{C})$ (infinite-dimensional general linear).

The quadratic elements with the basis $T^A{}_{,B}=\bar{a}^A a_B$, form a finite-dimensional subalgebra in $\mathrm{igl}(M|N;\mathbb{C})$, isomorphic to the superalgebra $\mathrm{gl}(M|N;\mathbb{C})$ (this explains our notation). The commutation relations in $\mathrm{igl}(M|N;\mathbb{C})$ are easily obtained by using (2.6), (2.19), and (2.14),

$$\begin{aligned}&[T^{A_1\cdots A_k}{}_{,B_1\cdots B_k},T^{C_1\cdots C_l}{}_{,D_1\cdots D_l}\}\\&\quad=\sum_{n,m}\frac{(-1)^m((k+l-n-m)!)^2}{n!\,m!\,(k-n)!\,(k-m)!\,(l-n)!\,(l-m)!}\,\delta(|n+m+1|_2)\\&\qquad\times\mathrm{Sym}((-1)^{\varepsilon^{A_k\cdots A_{k-m+1}}_{B_1\cdots B_{k-n}}\varepsilon^{C_{n+1}\cdots C_l}_{D_1\cdots D_m}}\,\delta^{A_k\cdots A_{k-m+1}}_{D_1\cdots D_m}\\&\qquad\times\delta^{C_1\cdots C_n}_{B_k\cdots B_{k-n+1}}\,T^{A_1\cdots A_{k-m}C_{n+1}\cdots C_l}{}_{,B_1\cdots B_{k-n}D_{m+1}\cdots D_l}),\end{aligned} \tag{2.23}$$

$$\delta^{A_1\cdots A_m}_{D_1\cdots D_m}=\delta^{A_1}_{D_1}\cdots\delta^{A_m}_{D_m}. \tag{2.24}$$

The commutation relations of $\mathrm{gl}(M|N;\mathbb{C})$,

$$[T^A{}_{,B},T^C{}_{,D}\}=\delta^C_B T^A{}_{,D}-(-1)^{\varepsilon^A_B\varepsilon^C_D}\,\delta^A_D T^C{}_{,B} \tag{2.25}$$

follow from the general formula (2.23). Thus we have constructed the infinite-dimensional generalization of the series of superalgebras $\mathrm{gl}(M|N;\mathbb{C})$.

If we introduce the Hermitian conjugation,

$$(a_A)^+=\bar{a}^A,\qquad(\bar{a}^A)^+=a_A \tag{2.26}$$

and extract the real form of $\mathrm{igl}(M|N;\mathbb{C})$ according to the rule

$$A^+=-A,\qquad A\in\mathrm{igl}(M|N;\mathbb{C}), \tag{2.27}$$

we obtain an algebra $\mathrm{iu}(M|N;\mathbb{C})$.

Let us now construct the infinite-dimensional generalization of the superalgebra sl($M|N$; $\mathbb{C}$), which is simple when $M \neq N$. The superalgebra sl($M|N$; $\mathbb{C}$) is formed by the elements having the null supertrace,

$$\tilde{\tilde{T}}_{A,}{}^{B} = a_A \bar{a}^B - \frac{1}{M-N}\delta^B_A(\bar{a}\cdot a),$$
$$\mathrm{str}(\tilde{\tilde{T}}_{A,}{}^{B}) = \sum_A (-1)^{\varepsilon_A}\tilde{\tilde{T}}_{A,}{}^{A} = 0. \tag{2.28}$$

In the case of infinite-dimensional algebras the situation is a little more complicated. To proceed further we need to introduce some notations and definitions. The infinite-dimensional Lie superalgebra is called:

(a) simple, if it does not contain ideals besides itself and zero;

(b) solvable, if it has the sequence of ideals

$$g = g_0 \supset g_1 \supset \cdots, \tag{2.29}$$

such that the factoralgebra $g_k/g_{k+1}$ are commutative. In the finite-dimensional case the sequence of ideals (2.29) can be chosen to be finite, which generally speaking cannot be done in the infinite-dimensional case.

The maximal solvable ideal, in analogy to the finite-dimensionnal case, is called the radical $R$ of algebra $g$ ($R(g)$). However, in the infinite-dimensional case the radical is not always unique. For every infinite-dimensional algebra $g$ the algebra $S = g/R(g)$ is semi simple.

A subalgebra $L \subset g$ is called a Levi subalgebra, if $g$ decomposes into the semi-direct sum

$$g = R(g) \oplus L\colon\ [L, L] \subset L, \quad [R, R] \subset R, \quad [R, L] \subset R. \tag{2.30}$$

The algebras $S$ and $L$ (if the latter exists and $R(g)$ is unique) are naturally isomorphic. In the finite-dimensional case there exists a Levi theorem on decomposition, which states that in any finite-dimensional Lie algebra on $\mathbb{R}$ or $\mathbb{C}$ there exists a Levi subalgebra. However this theorem generally speaking does not hold in the infinite-dimensional case. For the infinite-dimensional Lie algebra it is possible that the Levi subalgebra does not exist.

Bearing in mind all this let us consider the superalgebra igl($M|N$; $\mathbb{C}$) when $M \neq N$. Due to its construction this superalgebra contains a centre, generated by the elements of the form

$$T_n = (T)^n = (\bar{a}\cdot a)^n, \qquad n = 0, 1, 2, \ldots. \tag{2.31}$$

One can show, that in analogy to the decomposition

$$\mathrm{gl}(M|N; \mathbb{C}) = \mathrm{sl}(M|N; \mathbb{C}) \oplus \mathrm{gl}(1; \mathbb{C}), \tag{2.32}$$

there exists a decomposition of igl into the direct sum

$$\mathrm{igl}(M|N;\mathbb{C})=\mathrm{isl}^{\infty}(M|N;\mathbb{C})\oplus\left(\bigoplus_{n=0}^{\infty}\mathrm{gl}(1;\mathbb{C})_n\right), \tag{2.33}$$

where

$$\mathrm{isl}^{\infty}(M|N;\mathbb{C})=\mathrm{igl}(M|N;\mathbb{C})\Big/\left(\bigoplus_{n=0}^{\infty}\mathrm{gl}(1;\mathbb{C})_n\right) \tag{2.34}$$

and $\mathrm{gl}(1;\mathbb{C})_n$ are generated by $T_n$.

However, unlike the superalgebra $\mathrm{sl}(M|N;\mathbb{C})$, the superalgebra $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$ is not simple. We shall consider $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$ later, and now we shall prove the decomposition (2.33), i.e., the triviality of the central extension of $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$ to $\mathrm{igl}(M|N;\mathbb{C})$ by the elements $T_n$. Let us mention, that the superalgebra $\mathrm{igl}(M|N;\mathbb{C})$ has an invariant bilinear form

$$(A, B)=\mathrm{tr}(A * B), \tag{2.35}$$

$$\mathrm{tr}(A(Z))=A(0), \qquad Z=(\bar{a}^A, a_A), \tag{2.36}$$

where $A(Z)$, $B(Z)$ are the elements of the Grassmann shell of $\mathrm{igl}(M|N;\mathbb{C})$.

The invariance property of the bilinear form is

$$([A, B], C)+(A, [C, B])=0. \tag{2.37}$$

Triviality of the central extension of $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$ to $\mathrm{igl}(M|N;\mathbb{C})$ is equivalent to the condition

$$([A, B], T_n)=0 \tag{2.38}$$

for any $A$ and $B$.

By force of the relations (2.37) and the commutavity of $T_n$ with all elements of the $\mathrm{igl}(M|N;\mathbb{C})$, we have

$$([A, B], T_n)=-(A, [T_n, B])=0. \tag{2.39}$$

Therefore, the elements $T_n$ do not appear in the right-hand side of the commutators and the central extension is trivial.

Let us now consider the superalgebras $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$. The superalgebra $\mathrm{sl}(M|N;\mathbb{C})$ is a maximal finite-dimensional subalgebra of $\mathrm{isl}(M|N;\mathbb{C})$.

In order to find the radical of a superalgebra $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$, we decompose in into irreducible representation spaces with respect to $\mathrm{sl}(M|N;\mathbb{C})$. The corresponding basis can be chosen as follows:

$$ {}^{(n)}\tilde{\tilde{T}}_{A_1\cdots A_m,}{}^{B_1\cdots B_m} = \tilde{\tilde{T}}_{A_1\cdots A_m,}{}^{B_1\cdots B_m} * \underbrace{T * \cdots * T}_{n}, \tag{2.40} $$

$$ T = \bar{a}\cdot a, \qquad n = 0, 1, \ldots, m = 1, 2, \ldots, $$

$$ \sum_{A_m} \tilde{\tilde{T}}_{A_1\cdots A_{m-1}A_m,}{}^{A_m B_2\cdots B_m}(-1)^{\varepsilon_{A_m}} = 0. \tag{2.41} $$

Here $\tilde{\tilde{T}}_{A_1\cdots A_m,}{}^{B_1\cdots B_m}$ are the supertraceless parts of the elements of the form

$$ T_{A_1\cdots A_m,}{}^{B_1\cdots B_m} = \frac{1}{(m!)^2} a_{A_1}\cdots a_{A_m}\bar{a}^{B_1}\cdots\bar{a}^{B_m}. \tag{2.42} $$

In the superalgebra $\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})$ there exists a sequence of ideals

$$ T^n = \{{}^{(k)}\tilde{\tilde{T}}_{A_1\cdots A_m,}{}^{B_1\cdots B_m}, k = n, n+1, \ldots, m = 1, 2, \ldots\}. \tag{2.43} $$

From (2.20), (2.40), we have

$$ \begin{gathered} [T^n, T^m\} \subset T^{n+m}, \\ \mathrm{isl}^\infty(M\,|\,N;\mathbb{C}) = T^0 \supset T^1 \supset \cdots. \end{gathered} \tag{2.44} $$

The maximal solvable ideal in this sequence is $T^1$.

Let us consider the set of the elements[5] $T^0\backslash T^1$. This set is not a subalgebra. It can be proven by calculating the supertrace with respect to a pair of indices of the commutator

$$ [\xi_1, \xi_2\}^{A_1\cdots A_m}{}_{,B_1\cdots B_m} \qquad \text{at} \quad m > 1, \tag{2.45} $$

where

$$ \xi_{1(2)} = \sum_m \xi_{1(2)}^{A_1\cdots A_m}{}_{,B_1\cdots B_m}\, {}^{(0)}\tilde{\tilde{T}}_{A_1\cdots A_m,}{}^{B_1\cdots B_m}. $$

While the decomposition coefficients of $\xi_{1(2)}$ are supertraceless, the coefficients of $[\xi_1, \xi_2\}$ are not. The commutation relations $\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})$ are

$$ [{}^{(n)}\tilde{\tilde{T}}_A, {}^{(m)}\tilde{\tilde{T}}_B\} = \sum_k f^C_{A,B}(n, m, n+m+k)\,{}^{(n+m+k)}\tilde{\tilde{T}}_C, \tag{2.46} $$

where $A(B)$ are the collection indices of ${}^{(n)}\tilde{\tilde{T}}$. In the right-hand side of the commutator (2.46) at $n = m = 0$ are the elements ${}^{(k)}\tilde{\tilde{T}}$ with $k \geqslant 0$.

The set $T^0\backslash T^1$ does not contain nonzero ideals. Thus the ideal $T^1$ is a radical of $\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})$ and a set $\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})\backslash R(\mathrm{isl}^\infty(M\,|\,N;\mathbb{C}))$ is not a subalgebra is $\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})$. The simple algebra,

$$ \mathrm{isl}(M\,|\,N;\mathbb{C}) = \mathrm{isl}^\infty(M\,|\,N;\mathbb{C})/R(\mathrm{isl}^\infty(M\,|\,N;\mathbb{C})) \tag{2.47} $$

is a infinite-dimensional generalization of a simple algebra $\mathrm{sl}(M\,|\,N;\mathbb{C})$ $(N \neq M)$.

[5] The set $A\backslash B$, where $B \subset A$, consist of the elements $a \in A$ but $a \notin B$.

When $M=N$ the radical also contains, along with the elements of the form $T=\bar{a}a$, elements of the form

$$a\cdot\bar{a}=\sum_A(-1)^{\varepsilon_A}\,\bar{a}^A a_A. \tag{2.48}$$

However, here we shall not discuss this case in detail.

The superalgebra $\mathrm{isl}^\infty(M|N;\mathbb{C})$ contains a subalgebra of loop algebra for $\mathrm{sl}(M|N;\mathbb{C})$,

$$[{}^{(n)}T_A, {}^{(m)}T_B\}=f^C_{A,B}\,{}^{(n+m)}T_C, \qquad n, m=0, 1, ..., \tag{2.49}$$

where $A$, $B$, $C$ are the indices in the adjoint representation of $\mathrm{sl}(M|N;\mathbb{C})$ and $f^C_{A,B}$ are the structure constants of sl.

Let us notice that there exists a contraction of the superalgebra $\mathrm{isl}^\infty(M|N;\mathbb{C})$ to a subalgebra of the loop algebra of $\mathrm{isl}(M|N;\mathbb{C})$ with generators ${}^{(n)}T_A$, $n\geqslant 0$. This contraction is obtained by taking the limit $\lambda\to 0$ in the commutation relations (2.46) in the basis

$$\underset{\lambda}{{}^{(n)}\tilde{\tilde{T}}}{}_{A_1\cdots A_m}{}^{B_1\cdots B_m}=\lambda^{-n}\,{}^{(n)}\tilde{\tilde{T}}{}_{A_1\cdots A_m}{}^{B_1\cdots B_m}. \tag{2.50}$$

In conclusion it should be mentioned that $T^1$ is not the unique radical in $\mathrm{isl}^\infty(M|N;\mathbb{C})$. There exists a whole one-parameter family of radicals $T^1_\rho$ which are generated by elements $\bar{a}\cdot a-\rho\mathbb{1}$, where $\rho\in\mathbb{C}$, and $\mathbb{1}$ is the unit. Correspondingly, we have a one-parameter family of factoralgebras $\mathrm{isl}_\rho(M|N;\mathbb{C})$, and evidently $\mathrm{isl}(M|N;\mathbb{C})\simeq\mathrm{isl}_0(M|N;\mathbb{C})$.

## 3. Operator Realization of the Superconformal Algebra SU(2, 2 | 1)

In this section we consider an operatorial realization of SU(2, 2 | 1) that is convenient for our purposes in terms of two-component spinors.

The generating elements can be conveniently chosen as follows[6]:

$$a=\begin{pmatrix}a^\alpha\\ a_{\dot\beta}\end{pmatrix}, \qquad \bar{a}=(\bar{a}_\alpha, \bar{a}^{\dot\beta}), \qquad \varepsilon(a)=\varepsilon(\bar{a})=0, \tag{3.1}$$

$$[a^\alpha, \bar{a}_\beta]=2\delta^\alpha_\beta, \qquad [a_{\dot\alpha}, \bar{a}^{\dot\beta}]=2\delta^{\dot\beta}_{\dot\alpha}, \tag{3.2}$$

$$\{\alpha, \alpha^+\}=2, \qquad \varepsilon(\alpha)=\varepsilon(\alpha^+)=1. \tag{3.3}$$

The Hermitian conjugation is defined by

$$\begin{aligned}&(a^\alpha)^+=\bar{a}^{\dot\alpha}, \qquad (\bar{a}_\alpha)^+=a_{\dot\alpha}, \qquad (a_{\dot\alpha})^+=\bar{a}_\alpha,\\ &(\bar{a}^{\dot\alpha})^+=a^\alpha, \qquad (\alpha^+)^+=\alpha, \qquad (\alpha)^+=\alpha^+\end{aligned} \tag{3.4}$$

In formulas (3.1)–(3.4), $a$ is a Dirac spinor, $\bar{a}=a^+\gamma_0$ is a Dirac conjugate spinor, and $\alpha$ and $\alpha^+$ are the generating elements of the Clifford algebra.

[6] Let us mention, that the sets of generating elements $(a^\alpha, a_{\dot\beta}, \alpha)$ and $(\bar{a}_\alpha, \bar{a}^{\dot\alpha}, \alpha^+)$ are the supertwistor and its dual.

Let $(a, \bar{a}, \alpha, \alpha^+)$ now be the symbols of the operators in (3.1)–(3.4) (in the following we work only with symbols). The multiplication of the Weyl symbols is given by the formula (2.6), where operator (2.7) in this case has the form[7]

$$\vec{\Delta} = \bar{\partial}_\alpha \vec{\partial}^\alpha + \bar{\partial}^\beta \vec{\partial}_\beta - \bar{\partial}^\alpha \vec{\partial}_\alpha - \bar{\partial}_\beta \vec{\partial}^\beta + \frac{\overleftarrow{\partial}}{\partial\alpha}\frac{\vec{\partial}}{\partial\alpha^+} + \frac{\overleftarrow{\partial}}{\partial\alpha^+}\frac{\vec{\partial}}{\partial\alpha}. \tag{3.5}$$

If at least one of the symbols $A$ or $B$ is of the second order with respect to generating elements, the commutator (2.14) reduces to the Poisson bracket

$$[A, B\} = A\,\vec{\Delta}B - (-1)^{\varepsilon(A)\,\varepsilon(B)} B\,\vec{\Delta}A = 2A\,\vec{\Delta}B. \tag{3.6}$$

The symbol of the "particle number" operator in this case has the form

$$T = \bar{a}\cdot a + \alpha^+\alpha = \bar{a}_\alpha a^\alpha + \bar{a}^\beta a_\beta + \alpha^+\alpha. \tag{3.7}$$

The symbols of all generators, commuting with $T$, are presentable as linear combinations of basis symbols:

$$2M_{\alpha(2)} = -\bar{a}_\alpha a_\alpha, \qquad 2M_{\beta(2)} = \bar{a}_\beta a_\beta, \tag{3.8a}$$

$$2iP_{\alpha\beta} = \bar{a}_\beta a_\alpha, \qquad 2iK_{\alpha\beta} = \bar{a}_\alpha a_\beta, \tag{3.8b}$$

$$4D = \bar{a}\gamma_5 a = \bar{a}_\alpha a^\alpha - \bar{a}^\beta a_\beta, \tag{3.8c}$$

$$2Q_\alpha = a_\alpha \alpha^+, \qquad 2Q_\beta = \bar{a}_\beta\alpha, \qquad 2S_\alpha = \bar{a}_\alpha\alpha, \qquad 2S_\beta = a_\beta\alpha^+, \tag{3.8d}$$

$$2U = \frac{i}{4}\,\bar{a}\cdot a + i\alpha^+\alpha. \tag{3.8e}$$

These symbols generate an algebra SU(2, 2 | 1) with nonzero commutators (3.6)

$$[M_{\alpha(2)}, M_{\gamma(2)}] = 2\varepsilon_{\alpha\gamma} M_{\alpha\gamma}, \qquad [M_{\dot\alpha(2)}, M_{\dot\beta(2)}] = 2\varepsilon_{\dot\alpha\dot\beta} M_{\dot\alpha\dot\beta}, \tag{3.9a}$$

$$\left[M_{\alpha(2)}, \begin{pmatrix} P_{\gamma\beta} \\ K_{\gamma\beta} \end{pmatrix}\right] = \varepsilon_{\alpha\gamma}\begin{pmatrix} P_{\alpha\beta} \\ K_{\alpha\beta} \end{pmatrix}, \qquad \left[M_{\beta(2)}, \begin{pmatrix} P_{\gamma\rho} \\ K_{\gamma\rho} \end{pmatrix}\right] = \varepsilon_{\beta\rho}\begin{pmatrix} P_{\gamma\beta} \\ K_{\gamma\beta} \end{pmatrix}, \tag{3.9b}$$

$$[P_{\alpha\beta}, K_{\gamma\rho}] = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\rho} D + \varepsilon_{\alpha\gamma} M_{\beta\rho} + \varepsilon_{\beta\rho} M_{\alpha\gamma}, \tag{3.9c}$$

$$\left[D, \begin{pmatrix} P_{\alpha\beta} \\ K_{\alpha\beta} \end{pmatrix}\right] = \begin{pmatrix} -P_{\alpha\beta} \\ K_{\alpha\beta} \end{pmatrix}, \tag{3.9d}$$

$$\{Q_\alpha, Q_\beta\} = iP_{\alpha\beta}, \qquad \{S_\alpha, S_\beta\} = iK_{\alpha\beta}, \tag{3.9e}$$

$$\{Q_\alpha, S_\beta\} = -(M_{\alpha\beta} + \varepsilon_{\alpha\beta}(\tfrac{1}{2}D - iU)), \tag{3.9f}$$

$$\{Q_{\dot\alpha}, S_{\dot\beta}\} = (M_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}(\tfrac{1}{2}D + U)), \tag{3.9g}$$

$$[D, Q_{\binom{\alpha}{\beta}}] = -\tfrac{1}{2} Q_{\binom{\alpha}{\beta}}, \qquad [D, S_{\binom{\alpha}{\beta}}] = \tfrac{1}{2} S_{\binom{\alpha}{\beta}}, \tag{3.9h}$$

[7] The derivatives are defined by the relations $\partial_\alpha = \partial/\partial a^\alpha$, $\bar{\partial}^\alpha = \partial/\partial\bar{a}_\alpha$, $\partial^\beta = \partial/\partial a_\beta$, $\bar{\partial}_\beta = \partial/\partial\bar{a}^\beta$, $\partial_\alpha a^\gamma = \delta^\gamma_\alpha$, $\bar{\partial}^\alpha \bar{a}_\gamma = \delta^\alpha_\gamma$, $\partial^\beta a_\rho = \delta^\beta_\rho$, $\bar{\partial}_\beta \bar{a}^\rho = \delta^\rho_\beta$.

$$[U, Q_{\binom{\alpha}{\dot\beta}}] = \pm\tfrac{3}{4} i Q_{\binom{\alpha}{\dot\beta}}, \qquad [U, S_{\binom{\alpha}{\dot\beta}}] = \mp\tfrac{3}{4} i S_{\binom{\alpha}{\dot\beta}}, \tag{3.9i}$$

$$[P_{\alpha\dot\beta}, S_\gamma] = i\varepsilon_{\alpha\gamma} Q_{\dot\beta}, \qquad [P_{\alpha\dot\beta}, S_{\dot\rho}] = -i\varepsilon_{\dot\beta\dot\rho} Q_\alpha, \tag{3.9j}$$

$$[K_{\alpha\dot\beta}, Q_\gamma] = i\varepsilon_{\alpha\gamma} S_{\dot\beta}, \qquad [K_{\alpha\dot\beta}, Q_{\dot\rho}] = -i\varepsilon_{\dot\beta\dot\rho} S_\alpha. \tag{3.9k}$$

Evidently, $(P, K, M, D, U, Q, S)$ are the generators of translations, conformal boosts, Lorentz transformations, dilatations, chiral $U(1)$ transformations, supersymmetries, and special conformal supersymmetries.

The differential operators of the form

$$\tilde{A} = 2A\vec{\Delta}, \tag{3.10}$$

where $A$ are the symbols (3.8), form an adjoint representation of SU(2, 2 | 1). The explicit expressions for these operators are

$$i\tilde{P}_{\alpha\dot\beta} = \bar{a}_{\dot\beta}\bar{\partial}_\alpha - a_\alpha \partial_{\dot\beta}, \qquad i\tilde{K}_{\alpha\dot\beta} = \bar{a}_\alpha \bar{\partial}_{\dot\beta} - a_{\dot\beta}\partial_\alpha, \tag{3.11a}$$

$$\tilde{M}_{\alpha(2)} = a_\alpha \partial_\alpha - \bar{a}_\alpha \bar{\partial}_\alpha, \qquad \tilde{M}_{\dot\beta(2)} = \bar{a}_{\dot\beta}\bar{\partial}_{\dot\beta} - a_{\dot\beta}\partial_{\dot\beta}, \tag{3.11b}$$

$$2\tilde{D} = (\bar{a}_\alpha \bar{\partial}^\alpha - a^\alpha \partial_\alpha + a_{\dot\beta}\partial^{\dot\beta} - \bar{a}^{\dot\beta}\bar{\partial}_{\dot\beta}), \tag{3.11c}$$

$$\tilde{U} = \frac{i}{4}(\bar{a}_\alpha \bar{\partial}^\alpha - a^\alpha \partial_\alpha - a_{\dot\beta}\partial^{\dot\beta} + \bar{a}^{\dot\beta}\bar{\partial}_{\dot\beta}) + i\left(\alpha^+ \frac{\partial_l}{\partial\alpha^+} - \alpha\frac{\partial_l}{\partial\alpha}\right), \tag{3.11d}$$

$$\tilde{Q}_\alpha = \alpha^+ \bar{\partial}_\alpha + a_\alpha \frac{\partial_l}{\partial\alpha}, \qquad \tilde{Q}_{\dot\beta} = \bar{a}_{\dot\beta}\frac{\partial_l}{\partial\alpha^+} - \alpha\partial_{\dot\beta} \tag{3.11e}$$

$$\tilde{S}_\alpha = \bar{a}_\alpha \frac{\partial_l}{\partial\alpha^+} - \alpha\partial_\alpha, \qquad \tilde{S}_{\dot\beta} = \alpha^+\bar{\partial}_{\dot\beta} + a_{\dot\beta}\frac{\partial_l}{\partial\alpha}. \tag{3.11f}$$

From the commutation relations of SU(2, 2 | 1) we see, that in the basis (3.8) $\tilde{D}$ and $\tilde{U}$ are diagonal operators, as all generators have definite conformal $c$ and chiral $u$ weights:

$$\tilde{D}(A^{c,u}) = cA^{c,u}, \qquad \tilde{U}(A^{c,u}) = \tfrac{3}{2} iuA^{c,u}, \tag{3.12}$$

where $A^{c,u}$ are the basis elements (3.8).

The operators $\tilde{S}$ and $\tilde{Q}$ correspondingly raise and lower the conformal weight at $\frac{1}{2}$. The operators $\tilde{K}$ and $\tilde{P}$ correspondingly raise and lower the conformal weight at 1.

## 4. The Representations of SU(2, 2) with Higher Spins (Bose Case)

In this section we consider the representation of SU(2, 2) in the linear space $V$, which is the real form of aqpc(0 | 4; $\mathbb{C}$) (as linear space), obtained with the help of Hermitian conjugation (3.4).

The operators of representation of SU(2, 2) in $V$ act according to a formula

$$[A, T] = \tilde{A}(T), \qquad T \in V, \tag{4.1}$$

where $\tilde{A}$ are the operators (3.11).

Let us decompose $V$ into the irreducible representations spaces (irrepses) of SU(2, 2) and in each of them introduce the conformal basis, in which the irreps of SU(2, 2) are explicitly decomposed into the irrepses of the subalgebras

$$\mathrm{SU}(2, 2) \to \mathrm{sl}(2; \mathbb{C}) \oplus \mathrm{so}(1, 1), \tag{4.2}$$

where sl(2; ℂ) is a Lorentz algebra and so(1, 1) is generated by operator $\tilde{D}$. In such a basis all the generators (and the gauge fields corresponding to them) will have an explicit Lorenzian structure and definite conformal (Weyl) weight.

In order to decompose $V$ into the irrepses of SU(2, 2) in the space $V$ let us find all the highest vectors with a maximal conformal weight (as the operators from SU(2, 2) do not change the degree of homogeneity of the polynomials with respect to the generating elements, all the irrepses are finite-dimensional and, therefore, have a highest vector). The highest vectors of SU(2, 2) must satisfy the equalities

$$\tilde{K}_{\alpha\dot{\beta}}(T^s) = 0, \qquad \tilde{D}(T^s) = sT^s \tag{4.3}$$

($\tilde{K}$ is the raising operator).

The general solution of (4.3) for fixed $s = 0, 1, \ldots$ is a linear combination of vectors of a form

$${}^{(n)}T^s_{\alpha(s), \dot{\beta}(s)} = \bar{a}_{\alpha(s)} a_{\dot{\beta}(s)} (T)^n, \qquad n = 0, 1, \ldots, \tag{4.4}$$

where $T = \bar{a} \cdot a$ ($T = T(3.7)|_{\alpha = 0}$, because we consider purely Bose case).

Thus we see, that the space $V$ contains with the infinite multiplicity all the irrepses of SU(2, 2) with the highest weight $(s, s, 0)$, where in the signature $(s, l+j, l-j)$, $s$ is a highest conformal weight, and $(l, j)$—a Lorenzian signature of a highest vector with a maximal conformal weight.

The complete basis in the representation spaces is obtained from highest vectors (4.4) by the action of operators $\tilde{P}_{\alpha\dot{\beta}}$, which lower the conformal weight.

To obtain a sl(2; ℂ)-irreducible basis it is necessary to decompose the expression obtained into irreducible multispinors with the help of spinorial Clebsch–Gordan (C–G) coefficients (see Appendix B of this paper and the more detailed Appendix of the work [2]).

The resulting sl(2; ℂ)-irreducible conformal basis in a space $V$ has a form

$$\begin{aligned} {}^{(n)}T^{(s,c)}_{\alpha(2l), \dot{\beta}(2j)} &\sim \bar{C}_{\alpha(2l),}{}^{\rho(s), \gamma(s-c)} \bar{C}_{\dot{\beta}(2j),}{}^{\dot{\xi}(s), \dot{\delta}(s-c)} \\ &\times \underbrace{\tilde{P}_{\gamma\dot{\delta}} \cdots \tilde{P}_{\gamma\dot{\delta}}}_{s-c} \left({}^{(n)}T^s_{\rho(s), \dot{\xi}(s)}\right), \end{aligned} \tag{4.5}$$

where $\bar{C}$ are the spinorial C–G coefficients of sl(2; ℂ).

Performing the calculations in the right-hand side of (4.5), we obtain (details of the calculation are presented in Appendix D)

$$\begin{aligned} {}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)} &= \tfrac{1}{2} i^{c-s-1} \sum_{s_1+s_2=s+n} (-1)^{s_2-n-c/2} C^{(s+n-l+j+1)/2,(s+n+l-j+1)/2,s+1}_{(l+j+s_2-s_1+1)/2,(l+j+s_1-s_2+1)/2,l+j+1} \\ &\quad \times T^{(s_1,s_2,c/2,c/2)}_{\alpha(2l),\beta(2j)}, \quad s_1, s_2 = 0, \tfrac{1}{2}, 1, ..., \end{aligned} \tag{4.6}$$

where $C$ are the C–G coefficients of SU(2) and

$$\begin{aligned} T^{(s_1,s_2,c_1,c_2)}_{\alpha(2l),\beta(2j)} &= \frac{1}{\sqrt{(s_1+c_1)!\,(s_1-c_1)!\,(s_2+c_2)!\,(s_2-c_2)!}} \\ &\quad \times \bar{C}_{\alpha(2l),}{}^{\gamma(s_1+c_1),\rho(s_1-c_1)} \bar{C}_{\beta(2j),}{}^{\delta(s_2+c_2),\xi(s_2-c_2)} \\ &\quad \times \bar{a}_{\gamma(s_1+c_1)} a_{\rho(s_1-c_1)} \bar{a}_{\xi(s_2-c_2)} a_{\delta(s_2+c_2)}, \end{aligned} \tag{4.7}$$

$$T^{(s_1,s_2,c_1,c_2)}_{\alpha(2l),\beta(2j)} = T^{(s_1,c_1)}_{\alpha(2l)} \cdot \bar{T}^{(s_2,c_2)}_{\beta(2j)}. \tag{4.8}$$

Let us notice, that the multipliers in (4.8) are the generators of the two algebras shsc(1 | 3). In the expression (4.6) the normalization is chosen in such a way, that with respect to a bilinear form in $V$

$$(A, B) = \mathrm{tr}(A * B) = (A * B)(0), \tag{4.9}$$

the conformal basis is normalized

$$\begin{aligned} &\left({}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)},\ {}^{(n')}T^{(s',c')}_{\gamma(2l'),\rho(2j')}\right) \\ &\quad = \tfrac{1}{4}(-1)^{n+l+j+1}\, \delta_{n,n'} \delta_{l',l} \delta_{j,j'} \delta_{s,s'} \delta_{c,-c'} \varepsilon_{\alpha(2l),\gamma(2l)} \varepsilon_{\beta(2j),\rho(2j)}. \end{aligned} \tag{4.10}$$

The formula (4.10) follows from the orthogonality relations for C-G coefficients and from the orthonormality of the basis (4.7), (4.8) with respect to (4.9).

Therefore, we have introduced in $V$ the conformal basis ${}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)}$, where $s = 0, 1, ...$ defines a representation of SU(2, 2) with a signature $(s, s, 0)$, $n + 1 = 1, 2, ...$ numerates identical representations, $c = -s, -s+1, ..., s$ is a conformal weight of the generators and $l, j = 0, \frac{1}{2}, 1, ...$: $l, j \geqslant |c/2|$, $l + j \leqslant s$, $l + c/2$ and $j + c/2$ are integers.

The conformal basis (as a consequence of the index $(n)$) is defined nonunically. The transformations of the form

$${}^{(n)}\tilde{\bar{T}}^{(s,c)}_{\alpha(2l),\beta(2j)} = \sum_m C^{c,l,j}_{n,m}(s)\ {}^{(m)}T^{(s,c)}_{\alpha(2l),\beta(2j)} \tag{4.11}$$

with some matrix $C_{n,m}$ transforming one conformal basis into another.

The Hermitian conjugation (3.4) acts in the basis (4.9) according to the formula

$$\left({}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)}\right)^+ = -{}^{(n)}T^{(s,c)}_{\beta(2j),\dot{\alpha}(2l)}. \tag{4.12}$$

## 5. The Algebras $hsc^\infty(4)$ and $hsc(4)$

In this section we shall construct an algebra $isu^\infty(2,2)$, generalizing the conformal algebra SU(2, 2). The gauge fields of $isu^\infty(2,2)$ or $hsc^\infty(4)$, as we shall denote it in the following, have a form

$$\omega_\mu = \sum_{n,s,c,l,j} {}^{(n)}\omega_{\mu,}^{(s,c)\alpha(2l),\dot\beta(2j)}\, {}^{(n)}T^{(s,c)}_{\alpha(2l),\dot\beta(2j)}, \tag{5.1}$$

$$\omega_\mu^+ = -\omega_\mu. \tag{5.1a}$$

The summation parameters run through all possible range of values (see above, $s = 1, 2, ...$).

This gauge fields generalize the fields of the Weyl gravity for the case of higher spins.

The following terms correspond to the Weyl gravity (conformal multiplet of spin 2):

$$\begin{aligned}\omega_\mu^W &= \omega_{\mu,}^{(1,1)\alpha,\dot\beta} T^{(1,1)}_{\alpha,\dot\beta} + \omega_{\mu,}^{(1,0)\alpha(2)} T^{(1,0)}_{\alpha(2)} \\ &\quad + \omega_{\mu,}^{(1,0)\dot\beta(2)} T^{(1,0)}_{\dot\beta(2)} + \omega_\mu^{(1,0)} T^{(1,0)} + \omega_{\mu,}^{(1,-1)\alpha_{-1}\dot\beta} T^{(1,-1)}_{\alpha,\dot\beta},\end{aligned} \tag{5.2}$$

where we have omitted the index (0) in ${}^{(0)}\omega_\mu$ and ${}^{(0)}T$. Comparing the expression (4.6) for the generators $hsc^\infty(4)$ with Ex. (3.8) for the generators of conformal group, we obtain the relation

$$\begin{aligned}&(K_{\alpha\dot\beta}, M_{\alpha(2)}, M_{\dot\beta(2)}, -D, P_{\alpha\dot\beta}) \\ &\quad = (T^{(1,1)}_{\alpha,\dot\beta}, T^{(1,0)}_{\alpha(2)}, T^{(1,0)}_{\dot\beta(2)}, T^{(1,0)}, T^{(1,-1)}_{\alpha,\dot\beta})\end{aligned} \tag{5.3}$$

and, correspondingly, to the fields

$$\begin{aligned}&(f_{\mu,\alpha\dot\beta}, \omega_{\mu,\alpha(2)}, \omega_{\mu,\dot\beta(2)}, -b_\mu, e_{\mu,\alpha\dot\beta}) \\ &\quad = (\omega^{(1,1)}_{\mu,\alpha,\dot\beta}, \omega^{(1,0)}_{\mu,\alpha(2)}, \omega^{(1,0)}_{\mu,\dot\beta(2)}, \omega_\mu^{(1,0)}, \omega^{(1,-1)}_{\mu,\alpha,\dot\beta}).\end{aligned} \tag{5.4}$$

Now we can give explicit formulae for the curvatures of $hsc^\infty(4)$

$$R_{\mu\nu} = \partial_{[\mu}\omega_{\nu]} + [\omega_\mu, \omega_\nu], \tag{5.5a}$$

$$R^A_{\mu\nu} = \partial_{[\mu}\omega^A_{\nu]} + f^A{}_{BC}\,\omega^B_\mu \omega^C_\nu, \tag{5.5b}$$

where the commutator is defined by (2.14). The expresions for structure coefficients of $hsc^\infty(4)$ follow from the formulas for the structure coefficients of $shsc(1|3)$ and from expressions (4.6)–(4.8) for the conformal basis (see [1,2] and Appendix C of this paper).

The final expression for the curvature of $\mathrm{hsc}^{\infty}(4)$ are

$$
\begin{aligned}
{}^{(n)}R^{(s,c)}_{\mu\nu,\alpha(2l),\dot\beta(2j)} &= \partial^{(n)}_{[\mu}\,\omega^{(s,c)}_{\nu],\alpha(2l),\dot\beta(2j)} \\
&+\sum \delta(c'+c''-c)\,\delta(2m-l'-l''+l)\,\delta(2r-l''+l'-l)\,\delta(2t-l'+l''-l) \\
&\times \delta(2p-j'-j''+j)\,\delta(2q-j''-j+j')\,\delta(2k+j''-j-j') \\
&\times \delta(|s'+s''+s+n+n'+n''+1|_2) \\
&\times i^{s'+s''-s-1}\begin{bmatrix} n' & s' & c' & l' & j' \\ n'' & s'' & c'' & l'' & j'' \\ n & s & c & l & j \end{bmatrix} {}^{(n')}\omega^{(s',c')}_{\mu,\alpha(2t)\gamma(2m),\dot\beta(2k)\dot\rho(2p)}\;{}^{(n'')}\omega^{(s'',c'')\gamma(2m)}_{\nu,\alpha(2r)}{}_{,\dot\beta(2q)}{}^{\dot\rho(2p)},
\end{aligned}
\tag{5.6}
$$

where the set of structure coefficients takes the form

$$
\begin{bmatrix} n_1 & s_1 & c_1 & l_1 & j_1 \\ n_2 & s_2 & c_2 & l_2 & j_2 \\ n_3 & s_3 & c_3 & l_3 & j_3 \end{bmatrix}
= \sum_{s_i'+s_i''=s_i+n_i} \left\{ \prod_{i=1}^{3} C^{(s_i+n_i-l_i+j_i+1)/2,\,(s_i+n_i+l_i-j_i+1)/2,\,s_i+1}_{(l_i+j_i+s_i''-s_i'+1)/2,\,(l_i+j_i+s_i'-s_i''+1)/2,\,l_i+j_i+1} \right\}
\times \begin{pmatrix} s_1' & s_2' & s_3' \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} \\ l_1 & l_2 & l_3 \end{pmatrix}
\begin{pmatrix} s_1'' & s_2'' & s_3'' \\ \frac{c_1}{2} & \frac{c_2}{2} & \frac{c_3}{2} \\ j_1 & j_2 & j_3 \end{pmatrix},
\tag{5.7}
$$

$$s_i',\, s_i''=0,\, \tfrac{1}{2},\, 1,\, \ldots, \qquad i=1,2,3$$

and

$$
\begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_3 \\ l_1 & l_2 & l_3 \end{pmatrix}
$$

are the structure coefficients of $\mathrm{shsc}(1\,|\,3)$ (Appendix C).

In the derivation of (5.6) we used the symmetry property of the coefficients (5.7),

$$
\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = (-1)^{\sum_{i=1}^{3}(s_i+n_i+l_i+j_i)} \begin{bmatrix} D_2 \\ D_1 \\ D_3 \end{bmatrix}, \qquad D_i=(n_i,\, s_i,\, c_i,\, l_i,\, j_i),
\tag{5.8}
$$

which follow from the symmetry property of the structure coefficients of $\mathrm{shsc}(1\,|\,3)$

(C.6). Between the well-known expressions for the curvatures of conformal gravity [4] and our expressions (5.6) the following correspondence takes place:

$$(R_{\mu\nu,\alpha\beta}(K), R_{\mu\nu,\alpha(2)}(M), R_{\mu\nu,\beta(2)}(M), R_{\mu\nu}(D), R_{\mu\nu,\alpha\beta}(P)) \sim (R^{(1,1)}_{\mu\nu,\alpha,\beta}, R^{(1,0)}_{\mu\nu,\alpha(2)}, R^{(1,0)}_{\mu\nu,\beta(2)}, R^{(1,0)}_{\mu\nu}, R^{(1,-1)}_{\mu\nu,\alpha,\beta}). \tag{5.9}$$

Here it is assumed that in the correspondent expressions for hsc$^\infty$(4) curvature ${}^{(n)}\omega_\mu^{(s,c)}=0$ when $n>0$ or $s>1$. The algebra hsc$^\infty$(4), as its complexification isl$^\infty$(4; $\mathbb{C}$), is not simple. In Section 2 we considered the construction of simple algebras isl($M|N$; $\mathbb{C}$), which are the factoralgebras of isl$^\infty$($M|N$; $\mathbb{C}$) with respect to a radical. Here we present a detailed construction of a simple algebra hsc(4) $\sim$ isu(2, 2) = hsc$^\infty$(4)/$R$(hsc$^\infty$(4)). Let us introduce in hsc$^\infty$(4) a basis, analogous to that of (2.31) for the general complex case. The new generators take the form

$$ {}^{(n)}\tilde{\tilde{T}}^{(s,c)}_{\alpha(2l),\beta(2j)} = {}^{(0)}T^{(s,c)}_{\alpha(2l),\beta(2j)} * \underbrace{T * \cdots * T}_{n}, \tag{5.10}$$

$$T = \bar{a}\cdot a.$$

The sets $T^n = \{{}^{(k)}\tilde{\tilde{T}}^{(s,c)}_{\alpha(2l),\beta(2j)}, k \geqslant n\}$ form a sequence of solvable ideals (2.34), (2.35) in hsc$^\infty$(4). The ideal $T^1$ is a radical of hsc$^\infty$(4).

To calculate the structure coefficients in the new basis (5.10) it is sufficient to calculate the commutator $[{}^{(0)}T, {}^{(0)}T]$; i.e., due to the property $[A, B*C] = [A, B]*C + B*[A, C]$ and to the commutativity of $T$ with all the generators of hsc$^\infty$(4), the relation

$$[{}^{(n)}T, {}^{(m)}T] = [{}^{(0)}T, {}^{(0)}T] * \underbrace{T * \cdots * T}_{n+m} \tag{5.11}$$

takes place. The transformation to the new basis has the form

$$ {}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)} = \sum_{m=0}^{n} C_{n,m}(s)\, {}^{(m)}\tilde{\tilde{T}}^{(s,c)}_{\alpha(2l),\beta(2j)}, \tag{5.12}$$

where the expression for a matrix $C_{n,m}$ is given in Appendix E.

The curvatures of hsc$^\infty$(4) in the new basis take the form (5.6), where for the structure coefficients we have

$$\overline{\overline{\begin{bmatrix} n_1 & s_1 & c_1 & l_1 & j_1 \\ n_2 & s_2 & c_2 & l_2 & j_2 \\ n_3 & s_3 & c_3 & l_3 & j_3 \end{bmatrix}}} = \theta(n_3-n_1-n_2)\sum_m C_{m,n_3-n_1-n_2}(s_3) \begin{bmatrix} 0 & s_1 & c_1 & l_1 & j_1 \\ 0 & s_2 & c_2 & l_2 & j_2 \\ m & s_3 & c_3 & l_3 & j_3 \end{bmatrix}, \tag{5.13}$$

$$n_3-n_1-n_2 \leqslant m \leqslant s_1+s_2-s_3, \quad \theta(n)=1\ (0) \quad \text{when} \quad n\geqslant 0\ (n<0).$$

Now one can easily obtain a factoralgebra $\mathrm{hsc}(4)=\mathrm{hsc}^{\infty}(4)/T^1$. To obtain it, it is sufficient to demand

$$ {}^{(n)}\tilde{\tilde{T}}{}^{(s,c)}_{\alpha(2l),\beta(2j)}\equiv 0, \qquad n>0, \tag{5.14}$$

which is equivalent to $T^1\equiv 0$.

The gauge fields, which correspond to an algebra hsc(4), are of the form

$$\omega_\mu=\sum_{s,c,l,j}\omega_{\mu,}^{(s,c)\alpha(2l),}{}^{\beta(2j)}T^{(s,c)}_{\alpha(2l),\beta(2j)}. \tag{5.15}$$

The conformal multiplet for each spin $s=1,2,\ldots$ appears in the decomposition (5.15) only once—unlike the case of $\mathrm{hsc}^{\infty}(4)$.

The curvatures of hsc(4) have the form

$$\begin{aligned} R^{(s,c)}_{\mu\nu,\alpha(2l),\beta(2j)}&=\partial_{[\mu}\omega^{(s,c)}_{\nu],\alpha(2l),\beta(2j)}\\ &+\sum\delta(c'+c''-c)\,\delta(2m-l'-l''+l)\,\delta(2r-l''-l+l')\,\delta(2t-l'-l+l'')\\ &\times\delta(2p-j'-j''+j)\,\delta(2q-j+j'-j'')\,\delta(2k+j''-j'-j)\\ &\times\delta(|s'+s''-s+1|_2)(-1)^{(s'+s''-s-1)/2}\\ &\times\begin{bmatrix} s' & c' & l & j'\\ s'' & c'' & l'' & j''\\ s & c & l & j\end{bmatrix}\omega^{(s',c')}_{\mu,\alpha(2t)\gamma(2m),\beta(2k)\rho(2p)}\omega^{(s'',c'')\gamma(2m)}_{\nu,\alpha(2r)}{}_{,\beta(2q)}{}^{\rho(2p)}, \end{aligned}\tag{5.16}$$

$$\begin{aligned}\begin{bmatrix} s_1 & c_1 & l_1 & j_1\\ s_2 & c_2 & l_2 & j_2\\ s_3 & c_3 & l_3 & j_3\end{bmatrix}&=\sum_m\sqrt{\frac{(2m)!\,(2s_3+3)!}{(2s_3+2m+3)!}}\,\frac{(s_3+m+1)!}{m!\,(s_3+1)!}\\ &\times\begin{bmatrix} 0 & s_1 & c_1 & l_1 & j_1\\ 0 & s_2 & c_2 & l_2 & j_2\\ 2m & s_3 & c_3 & l_3 & j_3\end{bmatrix},\\ &0\leqslant m\leqslant(s_1+s_2-s_3)/2,\end{aligned}\tag{5.17}$$

where we have substituted into the formula for the structure coefficients (5.13) $(n'=n''=n=0)$ an explicit expression for $C_{2m,0}(s)$ (E. 14). The algebra hsc(4) contains SU(2, 2) as a maximal finite-dimensional subalgebra and the curvatures (5.16) are the direct generalizations of the curvatures of conformal gravity.

The generators corresponding to the conformal multiplets of spin one (the corresponding gauge fields have a spin two) form a subalgebra in $\mathrm{hsc}^{\infty}(4)$ with commutation relations

$$[{}^{(n)}\tilde{\tilde{T}}_A,{}^{(m)}\tilde{\tilde{T}}_B]=f^{C}_{AB}\,{}^{(n+m)}\tilde{\tilde{T}}_C, \tag{5.18}$$

where $A$, $B$, $C$ are the indices in the adjoint representation of SU(2, 2).

Let us notice, that there exists a contraction of the algebra $\mathrm{hsc}^{\infty}(4)$ to an algebra $\mathrm{hsc}(4)(\mathbb{R}[x])$ with ${}^{(n)}T_A$, $n \geqslant 0$ ($A$ is an index in hsc(4)). This contraction is obtained by taking the limit $\lambda \to 0$ in the commutation relations of $\mathrm{hsc}^{\infty}(4)$ in the basis

$$ {}^{(n)}_{\lambda}\tilde{\tilde{T}}^{(s,c)}_{\alpha(2l),\beta(2j)} = \lambda^{-n}\, {}^{(n)}\tilde{\tilde{T}}^{(s,c)}_{\alpha(2l),\beta(2j)}. \tag{5.19} $$

From the curvatures of algebra $\mathrm{hsc}^{\infty}(4)$ one can construct the exact invariant

$$ I = \int \mathrm{tr}(R \wedge R), \tag{5.20} $$

$$ I(\mathrm{hsc}^{\infty}(4)) = \sum_{\substack{n,s,c\\ l,j}} (-1)^{n+l+j+1} \int {}^{(n)}R^{(s,c)}_{\alpha(2l),\beta(2j)} \wedge {}^{(n)}R^{(s,\,-c)\alpha(2l),\,\beta(2j)} \tag{5.21} $$

(see (4.10)). However, the invariant (5.21) is the integral of the full derivative and does not generate a nontrivial dynamic.

In conclusion let us mention, that as the algebra $SU(2,2) \simeq SO(4,2)$ can be considered as an algebra of isometries of a space $\mathrm{adS}_5$ and the gauge fields of algebras $\mathrm{hsc}^{(\infty)}(4)$ in the corresponding basis can also be considered as the generalization on the case of higher spins of the $\mathrm{adS}_5$ gravity. The corresponding field theory will be considered by us separately.

## 6. The Representations of SU(2,2 | 1) with Higher Spins

In Section 4 we have constructed a conformal basis in linear space V. In this section we shall generalize those results on the supersymmetric case of a representation of $SU(2,2|1)$ in $SV$ (real form of the complex linear space $\mathrm{aqpc}(4|1;c)$). We decompose $SV$ into irrepses of $SU(2,2|1)$ and introduce in each of them the superconformal basis, connected with a reduction of an algebra to subalgebras

$$ SU(2,2|1) \to SU(2,2) \oplus u(1)_{\mathrm{chir}} \to \mathrm{sl}(2;\mathbb{C}) \oplus \mathrm{so}(1,1)_{\mathrm{conf}} \oplus u(1)_{\mathrm{chir}}, \tag{6.1} $$

where $u(1)_{\mathrm{chir}}$ is generated by the operator $\tilde{U}$ (3.11d) and $\mathrm{so}(1,1)_{\mathrm{conf}}$ is generated by $\tilde{D}$ (3.11c).

All the conformal highest vectors of representations of $SU(2,2|1)$ satisfy the relations

$$ \tilde{S}_{\alpha}(T^s) = 0, \qquad \tilde{S}_{\dot{\alpha}}(T^s) = 0, \qquad \tilde{D}(T^s) = sT^s, \tag{6.2} $$

where $\tilde{S}_{\alpha}$, $\tilde{S}_{\dot{\alpha}}$ are the raising operators (3.11f). The general solution of (6.2) is the same as that of (4.4).

All the irrepses of $SU(2,2|1)$ are obtained from the highest vectors (4.4) by the action of the lowering operators $\tilde{Q}_{\alpha}$, $\tilde{Q}_{\dot{\alpha}}$.

In order to decompose the irrepses of SU(2, 2 | 1) into the irrepses of SU(2, 2), let us find all the conformal highest vectors of SU(2, 2) satisfying (4.3). One can easily check, that together with (4.4), where $T$ is given by (3.7), the relation (4.3) in $SV$ is satisfied by the vectors

$$\tilde{Q}^{\alpha}({}^{(n)}T^{s}_{\alpha(s),\beta(s)}) \sim {}^{(n)}T^{s,s-1/2,1/2}_{\alpha(s-1),\beta(s)} = \bar{a}_{\alpha(s-1)} a_{\beta(s)}\, \alpha^{+} (T)^{n}, \tag{6.3a}$$

$$\tilde{Q}^{\beta}({}^{(n)}T^{s}_{\alpha(s),\beta(s)}) \sim {}^{(n)}T^{s,s-1/2,-1/2}_{\alpha(s),\beta(s-1)} = \bar{a}_{\alpha(s)} a_{\beta(s-1)}\, \alpha (T)^{n}, \tag{6.3b}$$

$$[\tilde{Q}^{\alpha}, \tilde{Q}^{\beta}]({}^{(n)}T^{s}_{\alpha(s),\beta(s)}) \sim {}^{(n)}T^{s,s-1,0}_{\alpha(s-1),\beta(s-1)}$$
$$= \bar{a}_{\alpha(s-1)} a_{\beta(s-1)} \left( \alpha^{+}\alpha + \frac{1}{2(s+1)}\, \bar{a}\cdot a \right) (T)^{n}, \tag{6.3c}$$

or, giving a general formula,

$${}^{(n)}T^{s,\sigma,u}_{\alpha(\sigma-u),\beta(\sigma+u)} = \bar{a}_{\alpha(\sigma-u)} a_{\beta(\sigma+u)}$$
$$\times (\alpha^{+})^{|u|+u} (\alpha)^{|u|-u} \left( \alpha^{+}\alpha + \frac{1}{2(s+1)}\, \bar{a}\cdot a \right)^{s-\sigma-|u|} (T)^{n}, \tag{6.4}$$
$$T = \bar{a}\cdot a + \alpha^{+}\alpha.$$

Here $s$ is a highest conformal weight in the irreps of SU(2, 2 | 1), $\sigma$ is a highest conformal weight of an irreps of SU(2, 2), which is equal to $\sigma = s,\ s-\frac{1}{2},\ s-1$, $u$ is a chiral weight, $u = -\frac{1}{2}, 0, \frac{1}{2}$, and the numbers $u$ and $\sigma$ are simultaneously integers or half-integers.

Thus we have decomposed every irreps of SU(2, 2 | 1) with the highest vector $T^{s}_{\alpha(s),\beta(s)}$ into the irrepses of SU(2, 2),

$$V_{s} = (s, s, 0) \oplus (s-\tfrac{1}{2}, s-\tfrac{1}{2}, \tfrac{1}{2}) \oplus (s-\tfrac{1}{2}, s-\tfrac{1}{2}, -\tfrac{1}{2}) \oplus (s-1, s-1, 0),$$
$$s = 1, 2, \ldots . \tag{6.5}$$

Let us mention, that the signature $(s-\frac{1}{2}, s-\frac{1}{2}, \pm\frac{1}{2})$ defines the complex-conjugated representations.

The adjoint representation decomposes as follows:

$$(1, 1, 0) - (M, P, K, D), \qquad (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) - (S_{\alpha}, Q_{\beta}),$$
$$(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}) - (S_{\beta}, Q_{\alpha}), \qquad (0, 0, 0) - (U).$$

The gauge fields $\omega^{A}_{\mu}$, where $A$ is an index in the representation space (6.5), are the superconformal multiplet with a spin content $s+1, s+\frac{1}{2}, s$. As is known [4], for such supermultiplets, the number of degrees of freedom off and on the mass shell is equal to zero (the Fermi degrees of freedom are taken with the minus sign), i.e., fulfils the necessary condition of construction of a minimal Lagrangian with a closed supersymmetry algebra.

In order to construct a complete basis in the irreducible representation spaces of SU(2, 2|1), it is necessary to

(1) act on (6.4) by the operators $\tilde{P}_{\alpha\dot\beta}$;

(2) decompose the thus-obtained expression into the irreducible multispinors.

The resulting superconformal basis has the form

$$\begin{aligned}{}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)} \sim \bar{C}_{\alpha(2l),}{}^{\rho(\sigma-u),\gamma(\sigma-c)}\bar{C}_{\dot\beta(2j),}{}^{\dot\xi(\sigma+u),\dot\delta(\sigma-c)} \\ \times \underbrace{\tilde{P}_{\gamma\dot\delta}\cdots\tilde{P}_{\gamma\dot\delta}}_{\sigma-c}\left({}^{(n)}T^{s,\sigma,u}_{\rho(\sigma-u),\dot\xi(\sigma+u)}\right),\end{aligned} \tag{6.6}$$

where $\bar{C}$ are the spinorial C-G coefficients of sl(2; $\mathbb{C}$).

Let us first calculate the expressions

$$\begin{aligned}{}^{(n)}P^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)} \sim \bar{C}_{\alpha(2l),}{}^{\rho(\sigma-u),\gamma(\sigma-c)}\bar{C}_{\dot\beta(2j),}{}^{\dot\xi(\sigma+u),\dot\delta(\sigma-c)} \\ \times \underbrace{\tilde{P}_{\gamma\dot\delta}\cdots\tilde{P}_{\gamma\dot\delta}}_{\sigma-c}\left({}^{(n)}P^{s,\sigma,u}_{\rho(\sigma-u),\dot\xi(\sigma+u)}\right),\end{aligned} \tag{6.7}$$

$$\begin{aligned}{}^{(n)}P^{s,\sigma,u}_{\rho(\sigma-u),\dot\xi(\sigma+u)} = \bar{a}_{\rho(\sigma-u)}a_{\dot\xi(\sigma+u)}(\alpha^+)^{s-\sigma+u} \\ \times(\alpha)^{s-\sigma-u}(\bar{a}_\alpha a^\alpha+\bar{a}^{\dot\beta}a_{\dot\beta})^n.\end{aligned} \tag{6.8}$$

Unlike the basis (6.6), in the basis (6.7) each vector $P$ belongs to the definite irreps of an algebra SU(2, 2), but not of a superalgebra SU(2, 2|1).

With the help of the formulas from Appendix D it is easy to find, in analogy to (4.6), the explicit expression for the basis (6.7),

$$\begin{aligned}{}^{(n)}P^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)} = \frac{1}{2}i^{c-\sigma-1}\sum_{\sigma_1+\sigma_2=\sigma+n}(-1)^{\sigma_2-n-(c+u)/2} \\ \times C^{(\sigma+n-l+j+1)/2,(\sigma+n+l-j+1)/2,\sigma+1}_{(\sigma_2-\sigma_1+l+j+1)/2,(\sigma_1-\sigma_2+l+j+1)/2,l+j+1} \\ \times(\alpha^+)^{s-\sigma+u}(\alpha)^{s-\sigma-u}T^{(\sigma_1,\sigma_2,(c-u)/2,(c+u)/2)}_{\alpha(2l),\dot\beta(2j)},\end{aligned} \tag{6.9}$$

where $T^{(\sigma_1,\sigma_2,c_1,c_2)}$ is defined in (4.7). Now we need to express the basis (6.6) in terms of the basis (6.9). Noting, that because of the grassmanian nature of $\alpha$ and $\alpha^+$ we have

$$(T)^n = (\bar{a}_\alpha a^\alpha+\bar{a}^{\dot\beta}a_{\dot\beta})^n + n\alpha^+\alpha(\bar{a}_\alpha a^\alpha+\bar{a}^{\dot\beta}a_{\dot\beta})^{n-1}, \tag{6.10}$$

and, taking into account,

$${}^{(n)}P^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)}(\bar{a}_\alpha a^\alpha+\bar{a}^{\dot\beta}a_{\dot\beta}) = \sqrt{(2\sigma+n+4)(n+1)}\;{}^{(n+1)}P^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)}, \tag{6.11}$$

as it follows from (6.9), we obtain the relations:

$$ {}^{(n)}T^{(s,s-1/2,c,u)}_{\alpha(2l),\beta(2j)} = {}^{(n)}P^{(s,s-1/2,c,u)}_{\alpha(2l),\beta(2j)}, \tag{6.12a} $$

$$ {}^{(n)}T^{(s,s,c,0)}_{\alpha(2l),\beta(2j)} = \sqrt{\frac{2s+n+3}{2s+3}}\, {}^{(n)}P^{(s,s,c,0)}_{\alpha(2l),\beta(2j)} + \sqrt{\frac{n}{2s+3}}\, {}^{(n-1)}P^{(s+1,s,c,0)}_{\alpha(2l),\beta(2j)}, \tag{6.12b} $$

$$ {}^{(n)}T^{(s,s-1,c,0)}_{\alpha(2l),\beta(2j)} = \sqrt{\frac{n+1}{2s+1}}\, {}^{(n+1)}P^{(s-1,s-1,c,0)}_{\alpha(2l),\beta(2j)} + \sqrt{\frac{2s+n+2}{2s+1}}\, {}^{(n)}P^{(s,s-1,c,0)}_{\alpha(2l),\beta(2j)}, \tag{6.12c} $$

$$ {}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)} = \sum_{n'+s'=n+s} C_\sigma(n,s,n',s')\, {}^{(n')}P^{(s',\sigma,c,u)}_{\alpha(2l),\beta(2j)}. \tag{6.12d} $$

The nonzero transformation coefficients in (6.12d) are

$$ C_\sigma(n,s,n,s) = \begin{cases} 1, & \sigma = s - \frac{1}{2} \\ \sqrt{\dfrac{2\sigma+n+3}{2\sigma+3}}, & \sigma = s, \end{cases} \tag{6.12e} $$

$$ C_s(n,s,n-1,s+1) = \sqrt{\frac{n}{2s+3}}, \qquad C_{s-1}(n,s,n+1,s-1) = \sqrt{\frac{n+1}{2s+1}}, $$

$$ C_{s-1}(n,s,n,s) = \sqrt{\frac{2s+n+2}{2s+1}}, $$

$$ C_\sigma(n,s,n',s') = C_\sigma(n',s',n,s). $$

For calculation of the structure coefficients of $\mathrm{shsc}^\infty(4|1)$ we shall need the inverse transform

$$ {}^{(n)}P^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)} = \sum_{s'+n'=s+n} C^{-1}_\sigma(n,s,n',s')\, {}^{(n')}T^{(s',\sigma,c,u)}_{\alpha(2l),\beta(2j)}, \tag{6.13a} $$

where the inverse matrix

$$ C^{-1}_\sigma(n,s,n',s') = (-1)^{s-s'}\, C_\sigma(n,s,n',s'). \tag{6.13b} $$

The Hermitian conjugation in the superconformal basis (6.12) is realized by

$$ \left({}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}\right)^+ = -{}^{(n)}T^{(s,\sigma,c,-u)}_{\beta(2j),\dot\alpha(2l)}. \tag{6.14} $$

We have therefore constructed a superconformal basis ${}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}$, where $s = 1, 2, \ldots$ determines the signature of the representations of $SU(2,2|1)$;

$n+1=1, 2, \ldots$ numerates the equivalent representations; $\sigma = s, s-\frac{1}{2}, s-1$ and the chiral weight $u=\frac{1}{2}$ ($\sigma + u$ is always an integer) determine the signature $(\sigma, \sigma, -u)$ of the representation of SU(2, 2) (see (6.5)); $c=-\sigma, -\sigma+1, \ldots, \sigma$ is a conformal weight of a generator; $(l, j)$, $l, j = 0, \frac{1}{2}, 1, \ldots$ is a Lorentzian signature such that

$$l \geqslant \left|\frac{c-u}{2}\right|, \qquad j \geqslant \left|\frac{c+u}{2}\right|, \qquad l+j \leqslant \sigma, \qquad l+\frac{c-u}{2}, \qquad j+\frac{c+u}{2}$$

are integers.

Let us mention, that the superconformal basis is defined modulo the transformations of a form

$$^{(n)}\tilde{\tilde{T}}^{(s,\sigma,c,u)}_{\alpha(2l),\dot{\beta}(2j)} = \sum_m C_{n,m}(s, \sigma, c, u, l, j)\, {}^{(m)}T^{(s,\sigma,c,u)}_{\alpha(2l),\dot{\beta}(2j)}. \tag{6.15}$$

## 7. The Superalgebras shsc$^\infty$(4 | 1) and shsc(4 | 1)

In this section we shall calculate the structure coefficients and obtain explicit expressions for the curvatures of superalgebras isu$^\infty$(2, 2 | 1) = shsc$^\infty$(4 | 1) and isu(2, 2 | 1) = shsc(4 | 1). These superalgebras are the supersymmetric extensions of the algebras hsc$^\infty$(4) and hsc(4).

The gauge fields of shsc$^\infty$(4 | 1) have the form

$$\omega_\mu = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \sum_{\substack{\sigma,c,u\\ l,j}} i^{-|2\sigma|_2}\, {}^{(n)}\omega_{\mu,}^{(s,\sigma,c,u)\alpha(2l),\dot{\beta}(2j)}\, {}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\dot{\beta}(2j)}. \tag{7.1}$$

The summation in (7.1) is performed over all possible values of parameters (see above). The grassmanian parity of the fields coincides with the grassmanian parity of corresponding generators and

$$\varepsilon(T^\sigma) = \varepsilon(\omega^\sigma_\mu) = 1(0) \text{ for half-integer(integer)}\, \sigma, \tag{7.2}$$

$$\omega^\sigma_\mu T^{\sigma'} = (-1)^{4\sigma\sigma'} T^{\sigma'} \omega^\sigma_\mu. \tag{7.3}$$

The decomposition (7.1) with infinite degeneracy contains all the conformal supermultiplets with the highest spin $s+1=2, 3, \ldots$ and the spin content $(s+1, s+\frac{1}{2}, s)$ (unity is added because of the vector index $\mu$).

The conformal supergravity supermultiplet is $(2, \frac{3}{2}, 1)$. Comparing (3.8d), (3.8e), and (6.12), (6.9) it is not difficult to find, that there exists a correspondence (see (5.3), (5.4))

$$\left(iT^{(1,1/2,1/2,1/2)}_{\dot{\alpha}}, iT^{(1,1/2,1/2,-1/2)}_{\alpha}, T^{(1,1/2,-1/2,1/2)}_{\alpha}, T^{(1,1/2,-1/2,-1/2)}_{\dot{\alpha}}, T^{(1,0,0,0)}\right)$$
$$= \left(S_{\dot{\alpha}}, S_\alpha, -Q_\alpha, Q_{\dot{\alpha}}, -\frac{2}{\sqrt{3}}U\right) \tag{7.4}$$

and analogously for the fields,

$$\left(\omega^{(1,1/2,1/2,1/2)}_{\mu,\dot\alpha},\ \omega^{(1,1/2,1/2,-1/2)}_{\mu,\alpha},\ -i\omega^{(1,1/2,-1/2,1/2)}_{\mu,\alpha},\ i\omega^{(1,1/2,-1/2,-1/2)}_{\mu,\dot\alpha},\ -\frac{2}{\sqrt{3}}\,\omega^{(1,0,0,0)}_{\mu}\right)$$
$$=(\phi_{\mu,\dot\alpha},\phi_{\mu,\alpha},\psi_{\mu,\alpha},\psi_{\mu,\dot\alpha},A_\mu), \tag{7.5}$$

where $\phi$, $\psi$, and $A$ are the connections of the special conformal supersymmetries, gravitino, and $u(1)$ gauge vector in usual conformal supergravity.

Now we can give explicit expressions for the curvatures of $\mathrm{shsc}^\infty(4\,|\,1)$. According to the general definition of the curvatures (5.5) and the formulas (6.9), (6.12) and Appendix C, we obtain

$$\begin{aligned}
{}^{(n)}R^{(s,\sigma,c,u)}_{\mu\nu,\alpha(2l),\dot\beta(2j)} &= \partial^{(n)}_{[\mu}\,\omega^{(s,\sigma,c,u)}_{\nu],\alpha(2l),\dot\beta(2j)}\\
&+\sum \delta(c'+c''-c)\,\delta(u'+u''-u)\,\delta(2m-l'-l''+l)\,\delta(2r-l''+l'-l)\\
&\times\delta(2t-l'+l''-l)\,\delta(2p-j'-j''+j)\,\delta(2q-j''-j+j')\,\delta(2k+j''-j-j')\\
&\times\delta(|s+s'+s''+n+n'+n''+1|_2)\,i^{\sigma'+\sigma''-\sigma-1}\\
&\times\begin{bmatrix} n' & s' & \sigma' & c' & u' & l' & j'\\ n'' & s'' & \sigma'' & c'' & u'' & l'' & j''\\ n & s & \sigma & c & u & l & j\end{bmatrix}\\
&\times {}^{(n')}\omega^{(s',\sigma',c',u')}_{\mu,\alpha(2t)\gamma(2m),\dot\beta(2k)\dot\rho(2p)}\ {}^{(n'')}\omega^{(s'',\sigma'',c'',u'')\gamma(2m)}_{\nu,\alpha(2r)}{}_{,\dot\beta(2q)},{}^{\dot\rho(2p)},
\end{aligned} \tag{7.6}$$

where the matrix of the structure coefficients has the form

$$\begin{bmatrix} n_1 & s_1 & \sigma_1 & c_1 & u_1 & l_1 & j_1\\ n_2 & s_2 & \sigma_2 & c_2 & u_2 & l_2 & j_2\\ n_3 & s_3 & \sigma_3 & c_3 & u_3 & l_3 & j_3\end{bmatrix} = \sum_{s'_i+n'_i=s_i+n_i}\ \sum_{\sigma'_i+\sigma''_i=\sigma_i+n'_i} (-1)^{s'_3-s_3}$$
$$\times\prod_{i=1}^{3}\left\{C_{\sigma_i}(s_i,n_i,s'_i,n'_i)\,C^{(\sigma_i+n'_i+j_i-l_i+1)/2,\,(\sigma_i+n'_i-j_i+l_i+1)/2,\,\sigma_i+1}_{(l_i+j_i+\sigma''_i-\sigma'_i+1)/2,\,(l_i+j_i-\sigma''_i+\sigma'_i+1)/2,\,l_i+j_i+1}\right\}$$
$$\times A^{s'_1-\sigma_1,s'_2-\sigma_2,s'_3-\sigma_3}_{u_1,u_2,u_3}\begin{pmatrix}\sigma'_1 & \sigma'_2 & \sigma'_3\\ \frac{c_1-u_1}{2} & \frac{c_2-u_2}{2} & \frac{c_3-u_3}{2}\\ l_1 & l_2 & l_3\end{pmatrix}\begin{pmatrix}\sigma''_1 & \sigma''_2 & \sigma''_3\\ \frac{c_1+u_1}{2} & \frac{c_2+u_2}{2} & \frac{c_3+u_3}{2}\\ j_1 & j_2 & j_3\end{pmatrix}, \tag{7.7}$$

$$s'_i,n'_i=0,1,\ldots,\quad \sigma'_i,\sigma''_i=0,\tfrac{1}{2},1,\ldots,\qquad i=1,2,3.$$

The $A$-coefficients are defined by the relation

$$(\alpha^+)^{l_1+u_1}\alpha^{l_1-u_1} * (\alpha^+)^{l_2+u_2}\alpha^{l_2-u_2}=\sum_{l_3,u_3} A^{l_1,l_2,l_3}_{u_1,u_2,u_3}(\alpha^+)^{l_3+u_3}\alpha^{l_3-u_3}. \tag{7.8}$$

By a simple calculation, we obtain

$$A^{0,0,0}_{0,0,0}=A^{1,1,0}_{0,0,0}=A^{0,1,1}_{0,0,0}=A^{1/2,1/2,0}_{1/2,-1/2,0}=A^{1/2,1/2,1}_{1/2,-1/2,0}=A^{1,1/2,1/2}_{0,1/2,1/2}=A^{0,1/2,1/2}_{0,1/2,1/2}=1, \quad (7.9a)$$

$$A^{l_2,l_1,l_3}_{u_2,u_1,u_3}=(-1)^{4l_1l_2+l_1+l_2-l_3}A^{l_1,l_2,l_3}_{u_1,u_2,u_3} \quad (7.9b)$$

$$A^{l_1,l_2,l_3}_{-u_1,-u_2,-u_3}=(-1)^{4l_1l_2+l_1+l_2-l_3}A^{l_1,l_2,l_3}_{u_1,u_2,u_3}, \quad (7.9c)$$

and all the $A$-coefficients besides (7.9a) and those obtainable from them, with the help of the symmetry properties (7.9b), (7.9c) are equal to zero.

From the symmetry properties of the $A$-coefficients (7.9b) and of the structure coefficients shsc(1 | 3) (C.6) a symmetry property of the coefficients (7.7) follows,

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}=(-1)^{\sum_{i=1}^{3}(s_i+n_i+l_i+j_i)+4\sigma_1\sigma_2+2\sigma_3}\begin{bmatrix} D_2 \\ D_1 \\ D_3 \end{bmatrix}, \quad (7.10)$$

where $D_i$ are the rows of table (7.7).

To construct a simple superalgebra,

$$\text{shsc}(4\,|\,1)=\text{shsc}^{\infty}(4\,|\,1)/R(\text{shsc}^{\infty}(4\,|\,1)), \quad (7.11)$$

let us introduce in shsc$^{\infty}(4\,|\,1)$ a basis, analogous to (5.10),

$$\begin{gathered} {}^{(n)}\tilde{\tilde{T}}^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}={}^{(0)}T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}*\underbrace{T*\cdots*T}_{n}, \\ T=\bar{a}\cdot a+\alpha^{+}\alpha. \end{gathered} \quad (7.12)$$

The matrix of the transformation from the new basis (7.12) to the old one (6.9), (6.12),

$${}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}=\sum_{m=0}^{n}\bar{C}_{n,m}(s)\;{}^{(m)}\tilde{\tilde{T}}^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)} \quad (7.13)$$

is calculated in Appendix E. The structure coefficients of shsc$^{\infty}(4\,|\,1)$ in the new basis can be written in the form

$$\begin{gathered} \widetilde{\widetilde{\begin{bmatrix} n_1 & s_1 & \sigma_1 & c_1 & u_1 & l_1 & j_1 \\ n_2 & s_2 & \sigma_2 & c_2 & u_2 & l_2 & j_2 \\ n_3 & s_3 & \sigma_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix}}} \\ =\theta(n_3-n_1-n_2)\sum_{m}\bar{C}_{m,n_3-n_1-n_2}(s)\begin{bmatrix} 0 & s_1 & \sigma_1 & c_1 & u_1 & l_1 & j_1 \\ 0 & s_2 & \sigma_2 & c_2 & u_2 & l_2 & j_2 \\ m & s_3 & \sigma_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix}, \\ n_3-n_1-n_2\leqslant m\leqslant s_1+s_2-s_3, \end{gathered} \quad (7.14)$$

where matrix $C$ is given in (E.15), (E.10).

In order to pass to the factoralgebra (7.11), let us demand

$$ {}^{(n)}\tilde{\tilde{T}}{}^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)} \equiv 0, \qquad n>0, \tag{7.15}$$

which is equivalent to $R(\mathrm{shsc}^{\infty}(4|1)) \equiv 0$.

The gauge field on shsc(4|1) has the form

$$\omega_\mu = \sum i^{-|2\sigma|_2} \omega_{\mu,}^{(s,\sigma,c,u)\alpha(2l),\beta(2j)} T^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)}. \tag{7.16}$$

The conformal supermultiplets of each spin occur in the decomposition (7.16) only once.

The curvatures of shsc(4|1) have the form

$$\begin{aligned} R^{(s,\sigma,c,u)}_{\mu\nu,\alpha(2l),\beta(2j)} &= \partial_{[\mu}\omega^{(s,\sigma,c,u)}_{\nu],\alpha(2l),\beta(2j)} \\ &+ \sum \delta(c'+c''-c)\,\delta(u'+u''-u)\,\delta(2m-l'-l''+l)\,\delta(2r-l''+l'-l) \\ &\times \delta(2t-l'+l''-l)\,\delta(2p-j'-j''+j)\,\delta(2q-j''-j+j')\,\delta(2k+j''-j-j') \\ &\times \delta(|s+s'+s''+1|_2)\, i^{\sigma'+\sigma''-\sigma-1} \\ &\times \begin{bmatrix} s' & \sigma' & c' & u' & l' & j' \\ s'' & \sigma'' & c'' & u'' & l'' & j'' \\ s & \sigma & c & u & l & j \end{bmatrix} \\ &\times \omega^{(s',\sigma',c',u')}_{\mu,\alpha(2t)\gamma(2m),\beta(2k)\dot\rho(2p)}\,\omega^{(s'',\sigma'',c'',u'')\gamma(2m)}_{\nu,\alpha(2r)}{}_{,\beta(2q),}{}^{\dot\rho(2p)}, \end{aligned} \tag{7.17}$$

where

$$\begin{bmatrix} s_1 & \sigma_1 & c_1 & u_1 & l_1 & j_1 \\ s_2 & \sigma_2 & c_2 & u_2 & l_2 & j_2 \\ s_3 & \sigma_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix} = \sum_m \sqrt{\frac{(2m)!\,(2s_3+2)!}{(2s_3+2m+2)!}}\,\frac{(s_3+\frac{3}{2})_m}{m!}$$

$$\times \begin{bmatrix} 0 & s_1 & \sigma_1 & c_1 & u_1 & l_1 & j_1 \\ 0 & s_2 & \sigma_2 & c_2 & u_2 & l_2 & j_2 \\ 2m & s_3 & \sigma_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix}, \qquad 0 \leqslant m \leqslant (s_1+s_2-s_3)/2. \tag{7.18}$$

The expressions (7.18) are obtained from the formula (7.14), in which one substitutes (E.17) at $n_1 = n_2 = n_3 = 0$.

The superalgebra shsc(4|1) contains SU(2, 2|1) as a maximal finite-dimensional subalgebra, and the curvatures of shsc(4|1) (7.17) are the generalizations of the curvatures of conformal supergravity.

Let us consider the Bose-subalgebra of the superalgebra $\mathrm{shsc}^{\infty}(4|1)$ and show, that an isomorphism take place,

$$(\mathrm{shsc}^{\infty}(4|1))_{\mathrm{B}} \simeq \mathrm{hsc}^{\infty}(4) \oplus \mathrm{hsc}^{\infty}(4) \oplus \left(\bigoplus_{n=0}^{\infty} u(1)_n\right). \tag{7.19}$$

We can choose the conformal basis in the $(\mathrm{shsc}^{\infty}(4|1))_B$ in the form

$$\begin{aligned} {}^{(n)}_{\pm}T^{(s,c)}_{\alpha(2l),\beta(2j)} &= \Pi_{\pm}\ {}^{(n)}P^{(s,s,c,0)}_{\alpha(2l),\beta(2j)}, \qquad s>0, \\ {}^{(n)}T &= {}^{(n)}T^{(1,0,0,0)} \qquad (\text{generators of } u(1)_n), \end{aligned} \tag{7.20}$$

where the symbols of the projection operators take the form

$$\Pi_{\pm} = \tfrac{1}{2}(1 \pm \alpha^{+}\alpha), \tag{7.21}$$

$$\Pi_{\pm} * \Pi_{\pm} = \Pi_{\pm}, \tag{7.22}$$

$$\Pi_{+} * \Pi_{-} = 0, \tag{7.23}$$

and $P$ is the basis (6.9).

In the basis (7.20) the isomorphism (7.19) becomes evident, because

$${}^{(n)}P^{(s,s,c,0)}_{\alpha(2l),\beta(2j)} = {}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)}, \qquad s>0, \tag{7.24}$$

where $T$ is the Bose generators (4.6) of $\mathrm{hsc}^{\infty}(4)$.

The maximal finite-dimensional subalgebra in $(\mathrm{shsc}(4|1))_B/\bigoplus_{n=0}^{\infty} u(1)_n$ coincides with the $SU(2,2)\oplus SU(2,2)$ and describes two "gravities." Analogous reducibility of the boson sector takes place in the case of $\mathrm{adS}_4$ superalgebras of higher spins and auxiliary fields shsa(1) in [17].

However, in the supercase each conformal multiplet with spin $s$ enters as a highest multiplet into the supermultiplet $(s, s-\frac{1}{2}, s-1)$ and as a lowest multiplet into the supermultiplet $(s+1, s+\frac{1}{2}, s)$. In particular, gauge fields with spin two enter the supermultiplets $(2, \frac{3}{2}, 1)$ and $(3, \frac{5}{2}, 2)$ and the supersymmetry transformations transform the second spin two into the spins 5/2 and 3. Thus, the superalgebra $\mathrm{shsc}^{\infty}(4|1)$ contains only one conformal superalgebra $SU(2,2|1)$.

It is extremely interesting to construct a theory with spontaneous symmetry breaking such that all the higher supermultiplets become massive and only the usual supergravity multiplet remains massless. This theory can possibly be based on the superalgebra $\mathrm{shsc}^{\infty}(4|1)$, in which each spin enters an infinite number of times and variants with a redistribution of degrees of freedom are possible.

In conclusion let us mention, that from the curvatures of $\mathrm{shsc}^{\infty}(4|1)$ one can build the topological invariant of type (5.20). It has the form

$$\begin{aligned} I(\mathrm{shsc}^{\infty}(4|1)) &= \sum A_{\sigma}(n,n',s,s')(-1)^{s+n-l-j} \\ &\times i^{-2(\sigma+1)} \int {}^{(n)}R^{(s,\sigma,c,u)}_{\alpha(2l),\beta(2j)} \wedge {}^{(n')}R^{(s',\sigma,-c,-u)\alpha(2l),\ \beta(2j)}, \end{aligned} \tag{7.25}$$

where the coefficients of a bilinear form are given by

$$A_{\sigma}(n,n',s,s') = \delta(n+s-n'-s') \sum_{s''+n''=s+n} C_{\sigma}(n,s,n'',s'')\, C_{\sigma}(n',s',n'',s''). \tag{7.26}$$

## 8. The Extended Conformal Superalgebras $\mathrm{shsc}^{\infty}(4|N)$ and $\mathrm{shsc}(4|N)$

Here we shall briefly discuss the extended conformal superalgebras. Their complexifications $\mathrm{isl}^{\infty}(4|N; \mathbb{C})$ and $\mathrm{isl}(4|N; \mathbb{C})$ were discussed in the Section 2. To construct the corresponding gauge theories it is necessary to find a superconformal basis connected with the reduction of an algebra to subalgebras,

$$\begin{aligned} \mathrm{SU}(2,2|N) &\to \mathrm{SU}(2,2) \oplus \mathrm{SU}(N) \oplus u(1)_{\mathrm{chir}} \\ &\to \mathrm{sl}(2;\mathbb{C}) \oplus \mathrm{so}(1,1)_{\mathrm{conf}} \oplus \mathrm{SU}(N) \oplus u(1)_{\mathrm{chir}}. \end{aligned} \tag{8.1}$$

To construct this basis one must take the generating elements (3.1), (3.2), and instead of the pair $\alpha$, $\alpha^+$ to introduce $2N$ generating elements

$$\{\alpha_i, \alpha_j^+\} = 2\delta_{ij}, \qquad i, j = 1, ..., N. \tag{8.2}$$

The symbols of operators which are quadratic with respect to all generating elements and commuting with the "particle number" operator form the superalgebra $\mathrm{U}(2,2|N)$.

The gauge fields of a superalgebra $\mathrm{shsc}^{\infty}(4|N)$ contain with an infinite degeneracy all $\mathrm{SU}(2,2|N)$ supermultiplets with highest spins $s = 2, 3, ....$ A superalgebra $\mathrm{shsc}(4|N)$, being a factoralgebra of $\mathrm{shsc}^{\infty}(4|N)$, generates all such supermultiplets only once. The corresponding curvatures generalize the curvatures of usual conformal supergravity with $N \leqslant 4$ and what is especially interesting, provide a possibility of construction of conformal supergravity with higher spins for $N > 4$.

The most interesting case is the case of $N = 5$ superalgebra. The corresponding theory includes the *grand unification group* SU(5). It is known, that one cannot construct a usual $N = 5$ conformal supergravity off the mass shell because of the necessity of introducing higher spins (spin $\frac{5}{2}$).

In our case we do not restrict ourselves to the case of spin two and, generally speaking, the restriction on $N$ does not appear. The construction of a conformally-invariant theory of higher spins with $N \geqslant 5$ is a very appealing problem, and especially a variant of such a theory with spontaneously broken conformal symmetry. In this case one can hope to obtain a unified theory, which includes Einstein supergravity, the fields of grand unification, and the massive higher spins.

However, the knowledge of superalgebra is not sufficient for the construction of such a theory. It is also necessary to construct the corresponding multiplets of lower spins and to find a complete set of auxiliary fields (the same holds for all theories with $N > 1$).

## 9. The $\mathrm{shsc}_{\rho}^{(n)}(4|N)$ Superalgebras

We have been considering complex superalgebras $\mathrm{isl}^{\infty}(M|N; \mathbb{C})$ and $\mathrm{isl}(M|N; \mathbb{C})$ and their real forms (for $M = 4$) $\mathrm{shsc}^{\infty}(4|N)$ and $\mathrm{shsc}(4|N)$.

The $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$ and $\mathrm{shsc}^{\infty}(4|N)$ superalgebras, as mentioned above, contain an infinite family of the ideals $T^n$, which are embedded in each other. Under the factorization with respect to the maximal ideal, we have obtained simple superalgebras $\mathrm{isl}(M|N;\mathbb{C})$ and $\mathrm{shsc}(4|N)$. However, can all factoralgebras be considered with respect to all the ideals $T^n$ as

$$\mathrm{isl}^{(n)}(M|N;\mathbb{C}) = \mathrm{isl}^{\infty}(M|N;\mathbb{C})/T^n, \tag{9.1}$$

$$\mathrm{shsc}^{(n)}(4|N) = \mathrm{shsc}^{\infty}(4|N)/T^n. \tag{9.2}$$

These factoralgebras are evidently obtained, under the identification

$${}^{(k)}\tilde{\tilde{T}}_{A(m),}{}^{B(m)} \equiv 0, \qquad k \geqslant n, \tag{9.3}$$

$${}^{(k)}\tilde{\tilde{T}}^{(s,\sigma,c,u)}_{\alpha(2l),\dot{\beta}(2j)} \equiv 0, \qquad k \geqslant n \ (N=1). \tag{9.4}$$

Therefore, we have a family of superalgebras which generalize the finite dimensional sl and SU superalgebras. The $\mathrm{isl}^{(n)}(M|N;\mathbb{C})$ superalgebras contain the $\mathrm{sl}(M|N;\mathbb{C})$ representations with multiplicity $n$ and the $\mathrm{shsc}^{(n)}(4|N)$ superalgebras contain conformal supermultiplets of all spins with multiplicities also equal to $n$. When $n=1$ we have, by definition, $\mathrm{isl}^{(1)}(M|N;\mathbb{C}) = \mathrm{isl}(M|N;\mathbb{C})$ and $\mathrm{shsc}^{(1)}(4|N) = \mathrm{shsc}(4|N)$.

In Section 2 we have mentioned that there exists a whole family of factoralgebras $\mathrm{isl}_{\rho}(M|N;\mathbb{C})$ because $T^1$ is not a unique radical in $\mathrm{isl}^{\infty}(M|N;\mathbb{C})$. Analogously to that there exists a family of factoralgebras $\mathrm{isl}_{\rho}^{(n)}(M|N;\mathbb{C})$ for each fixed $n$. Corresponding ideals $T^n_{\rho}$ are defined as in (2.43) where instead of $T=\bar{a}\cdot a$ in (2.40) there is $T_{\rho}=\bar{a}\cdot a-\rho\mathbb{1}$, $\rho\in\mathbb{C}$. Respectively, there exist real superalgebras $\mathrm{shsc}_{\rho}^{(n)}(4|N)$.

Consider maximal finite-dimensional subalgebras of $\mathrm{isl}^{(n)}(M|N;\mathbb{C})$ and $\mathrm{shsc}^{(n)}(4|N)$. There are furnished by the generators ${}^{(m)}\tilde{\tilde{T}}_A$, $m=0,1,\ldots,n-1$, $A$ being an index of the adjoint representation of $\mathrm{sl}(M|N;\mathbb{C})$ or $\mathrm{SU}(2,2|N)$.

The commutation relations in these superalgebras are read

$$[{}^{(k)}\tilde{\tilde{T}}_A, {}^{(m)}\tilde{\tilde{T}}_B\} = f^{C}_{A,B}\, {}^{(k+m)}\tilde{\tilde{T}}_C\, \theta(n-k-m), \tag{9.5}$$

where $f^C_{A,B}$ are the $\mathrm{sl}(M|N;\mathbb{C})$ or $\mathrm{SU}(2,2|N)$ structure constants. It is easy to verify (9.5) when one recalls the commutation relations of isl or shsc and one takes into account Eqs. (9.3), (9.4). A straightforward calculation shows that the relations (9.5) obey the Jacobi identities

$$\theta(n-k-l-m)(\theta(n-k-l)f^E_{D,C}f^C_{A,B}(-1)^{\varepsilon_B\varepsilon_D} + \theta(n-k-m)f^E_{B,C}f^C_{D,A}(-1)^{\varepsilon_B\varepsilon_A} + \theta(n-m-l)f^E_{A,C}f^C_{B,D}(-1)^{\varepsilon_A\varepsilon_D}) \equiv 0 \tag{9.6}$$

Note that the relations (9.5) define a family of superalgebras for some arbitrary initial superalgebra.

Gauge fields for the $\mathrm{shsc}^{(n)}(4|N)$ superalgebra contain $n$ spin two supermultiplets.

In conclusion, note that extensions similar to (9.5) exist also for other supergravity and higher spin superalgebras. The corresponding extension of supergravity describes a collection of massless spin $\frac{3}{2}$ and 2 fields analogously to the construction of Ref. [18].

## APPENDIX

### A. *Notations and Conventions*

We follow the conventions of Refs. [1–3, 11–14]. The two-component spinorial indices are raised and lowered by means of $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{12} = \varepsilon^{12} = 1$, as $A^\alpha = \varepsilon^{\alpha\beta} A_\beta$, $A_\beta = \varepsilon_{\alpha\beta} A^\alpha$, and analogously for dotted indices.

A symmetrization is implied for any set of upper or lower dotted or undotted spinorial indices denoted by like letters. The usual summation convention is understood for each pair of a lower and an upper index denoted by the same letter. We use the notations such as $A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}$,

$$\underbrace{A_{\alpha\cdots\alpha}}_{n} = A_{\alpha(n)}, \qquad \underbrace{\varepsilon_{\alpha\beta}\cdots\varepsilon_{\alpha\beta}}_{n} = \varepsilon_{\alpha(n),\beta(n)},$$
$$\underbrace{\delta^\gamma_\alpha\cdots\delta^\gamma_\alpha}_{n} = \delta^{\gamma(n)}_{\alpha(n)}, \qquad \underbrace{a_\alpha\cdots a_\alpha}_{n} = a_{\alpha(n)}, \qquad \underbrace{q_\alpha\cdots q_\alpha}_{n} = q_{\alpha(n)}, \qquad \text{etc.} \tag{A.1}$$

The four-dimensional world indices are $\nu$, $\mu = 0, 1, 2, 3$. The metric has the signature $(+, -, -, -)$.

For a change of notations from the Lorentz indices $(a, b, ...)$ to the spinorial ones, the matrices $\sigma_a^{\alpha\dot\beta} = (I, \sigma_1, \sigma_2, \sigma_3)$ are to be employed ($\sigma_1$, $\sigma_2$, and $\sigma_3$ are the Pauli matrices). We often use the notations

$$\delta(n) = 1(0), \qquad n = 0\ (n \neq 0),$$
$$\theta(n) = 1(0), \qquad n \geqslant 0\ (n < 0),$$
$$|n|_2 = 1(0), \qquad n = 2k+1(2k),\ n,\ k \in \mathbb{Z},$$
$$\underbrace{A_{\alpha\cdots\alpha}}_{n} = \frac{1}{n!}\,(A_{\alpha_1\cdots\alpha_n} + (n!-1)\text{ permutations of } \alpha_1\cdots\alpha_n), \tag{A.2}$$
$$A_{\alpha(n),}{}^{\alpha(m)} = A_{(\beta_1\cdots\beta_m\alpha_{m+1}\cdots\alpha_n),}{}^{(\beta_1\cdots\beta_m)} \qquad \text{at} \quad n \geqslant m,$$

where brackets denote full symmetrization.

### B. *The Spinorial Clebsch–Gordan (C–G) Coefficients*

A spinor $T_{\alpha(2l),\dot\beta(2l)}$ can be decomposed into irreducible symmetric multispinors according to [15],

$$T_{\alpha(2l),\beta(2l')} = \sum_{l''} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} T^{(l,l')}_{\gamma(2l'')}, \tag{B.1}$$
$$T^{(l,l')}_{\gamma(2l'')} = \bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} T_{\alpha(2l),\beta(2l')}, \tag{B.2}$$

where the spinorial C–G coefficients are

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = C(l, l', l'')\,\varepsilon_{\alpha(2u),\beta(2u)}\,\delta^{\gamma(2l'')}_{\alpha(2s)\beta(2t)}, \qquad \text{(B.3a)}$$

$$\bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} = C(l, l', l'')\,\varepsilon^{\alpha(2u),\beta(2u)}\,\delta^{\alpha(2s)\beta(2t)}_{\alpha(2l'')}, \qquad \text{(B.3b)}$$

$$C(l, l', l'') = \sqrt{\frac{(2l)!\,(2l')!\,(2l''+1)!}{(2u)!\,(2t)!\,(2s)!\,(l+l'+l''+1)!}}, \qquad \text{(B.4a)}$$

$$2u = l + l' - l'', \qquad 2s = l - l' + l'', \qquad 2t = l' - l + l''. \qquad \text{(B.4b)}$$

The symmetry properties are expressed as

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = (-1)^{l+l'-l''}\,C_{\beta(2l'),\alpha(2l),}{}^{\gamma(2l'')}, \qquad \text{(B.5)}$$

and the orthogonality properties as

$$\sum_{l''} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')}\,\bar{C}_{\gamma(2l''),}{}^{\delta(2l),\beta(2l')} = \delta^{\delta(2l)}_{\alpha(2l)}\,\delta^{\rho(2l')}_{\beta(2l')}, \qquad \text{(B.6)}$$

$$\bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')}\,C_{\alpha(2l),\beta(2l'),}{}^{\rho(2l'')} = \delta^{\rho(2l'')}_{\gamma(2l'')}. \qquad \text{(B.7)}$$

These formulae are analogous to the corresponding usual relations for $C^{l,l',l''}_{m,m',m''}$ [16]. The spinorial analog of the intertwining formula for five $C^{l,l',l''}_{m,m',m''}$ coefficients [16] is written

$$\bar{C}_{\alpha(2j),}{}^{\lambda(2j_1),\rho(2j_2)}\,\bar{C}_{\beta(2k),}{}^{\delta(2k_1)\varepsilon(2k_2)}\,C_{\lambda(2j_1),\delta(2k_1),}{}^{\xi(2j_1')}$$
$$\times\, C_{\rho(2j_2),\varepsilon(2k_2),}{}^{\zeta(2j_2')}\,C_{\xi(2j_1'),\zeta(2j_2'),}{}^{\gamma(2j')}$$
$$= \sqrt{(2j_1'+1)(2j_2'+1)(2j+1)(2k+1)}\begin{Bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j_1' & j_2' & j \end{Bmatrix} C_{\alpha(2j),\beta(2k),}{}^{\gamma(2j')}, \qquad \text{(B.8)}$$

where $\{\cdots\}$ are the $9j$-coefficients [16]. The triangle coefficient $\Delta(l, l', l'')$ is

$$\Delta(l, l', l'') = \frac{(l+l'-l'')!\,(l-l'+l'')!\,(l'+l''-l)!}{(l+l'+l''+1)!}. \qquad \text{(B.9)}$$

Note that the shs(1 | 2) commutation relations [6] can be rewritten as

$$[T_{\alpha(2l)}, T_{\beta(2l')}\} = \sum_{l''} \frac{i^{l+l'-l''-1}}{\sqrt{(2l''+1)\,\Delta(l, l', l'')}}$$
$$\times\, \delta(|4ll' + l + l' - l'' + 1|_2)\,C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')}\,T_{\gamma(2l'')}. \qquad \text{(B.10)}$$

All formulae of this appendix are true for dotted multispinors.

C. *Some Results of Refs.* [1, 2]

In this appendix for the reader's convenience we list some results, referring to the algebra shsc(1 | 3).

The generating elements in the operator realization of shsc(1 | 3) have the form

$$[a_\alpha, b_\beta] = 2i\varepsilon_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [b_\alpha, b_\beta] = 0, \qquad a_\alpha^+ = a_\alpha, \qquad b_\alpha^+ = b_\alpha. \tag{C.1}$$

The conformal basis in shsc(1 | 3) is

$$T^{(s,c)}_{\alpha(2l)} = \frac{1}{2i\sqrt{(s+c)!\,(s-c)!}}\,\bar{C}_{\alpha(2l),}{}^{\beta(s+c),\gamma(s-c)}\, b_{\beta(s+c)}\, a_{\gamma(s-c)}, \tag{C.2}$$

where $\bar{C}$ are the C–G coefficients (B.3), (B.4). The number $s = 0, \frac{1}{2}, 1, ...$ numerates the representations of so(3, 2) with a signature $(s, s)$, where $c = -s, -s+1, ..., s$ is the conformal weight of a generator and $l = |c|, |c|+1, ..., s$ is the signature of a representation of so(2, 1). The associative product of Weyl symbols (C.2) has, as it was shown in [1, 2], the form

$$\begin{aligned}(T^{(s,c)}_{\alpha(2l)} * T^{(s',c')}_{\beta(2l')}) = \tfrac{1}{2}\sum i^{s+s'-s''-1}\,&\delta(c+c'-c'')\\ &\times \delta(2u-l-l'+l'')\,\delta(2v-l+l'-l'')\,\delta(2t-l'+l-l'')\\ &\times \begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} \varepsilon_{\alpha(2u),\beta(2u)}\, T^{(s'',c'')}_{\alpha(2v)\beta(2t)}\end{aligned} \tag{C.3}$$

with the number coefficient

$$\begin{aligned}\begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} &= \sqrt{\frac{(2l+1)!\,(2l'+1)!\,(2l''+1)!}{(l+l'-l'')!\,(l+l''-l')!\,(l'+l''-l)!\,(l+l'+l''+1)!}}\\ &\times \sum_{k,k',k''} (-1)^{(s+s'-s''-k-k'+k'')/2}\\ &\times \frac{d^{l}_{c,k}\left(\frac{\pi}{2}\right) d^{l'}_{c',k'}\left(\frac{\pi}{2}\right) d^{l''}_{k'',c''}\left(-\frac{\pi}{2}\right)}{\sqrt{\Delta\left(\frac{s+k}{2}, \frac{s'+k'}{2}, \frac{s''+k''}{2}\right)\Delta\left(\frac{s-k}{2}, \frac{s'-k'}{2}, \frac{s''-k''}{2}\right)}}\\ &\times \begin{Bmatrix} \frac{s+k}{2}, & \frac{s-k}{2}, & l \\ \frac{s'+k'}{2}, & \frac{s'-k'}{2}, & l' \\ \frac{s''+k''}{2}, & \frac{s''-k''}{2}, & l'' \end{Bmatrix}.\end{aligned} \tag{C.4}$$

Here $\Delta(l, l', l'')$ are given by formula (B.9) and $k = -l, -l+1, ..., l$, and similarly for $k', k''$.

We have expressed the structure constants of the associative algebra aq$(0|4;\mathbb{C})$ through the nine $j$-coefficients and some particular values of the Wigner $d$-functions. The summation in (C.3) formally is

$$u, v, t, s'', l'' = 0, \tfrac{1}{2}, 1, \ldots; \qquad c'' = \ldots, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \ldots.$$

However, due to the extension of the definition of the coefficients (C.4) and $\delta$-functions, the region of the summation becomes restricted non-trivially.

We extend the definition of the coefficients (C.4) by putting them to zero if at least one of the following conditions is not satisfied:

$$\begin{gathered} s'' \in \{|s-s'|, \ldots, s+s'\}; \qquad l'' \in \{|l-l'|, \ldots, l+l'\}; \qquad c+c'-c''=0; \\ l \in \{|c|, \ldots, s\}; \qquad l' \in \{|c'|, \ldots, s'\}; \qquad l'' \in \{|c''|, \ldots, s''\}; \\ |c| \leqslant s; \qquad |c'| \leqslant s'; \qquad |c''| \leqslant s''. \end{gathered} \tag{C.5}$$

Note that due to the symmetry of the nine $j$-coefficients, the coefficients (C.4) satisfy the symmetry condition

$$(D, D', D'') = (-1)^{s+s'+s''+l+l'+l''}(D', D, D''), \tag{C.6}$$

where $D$ are the columns in the table (C.4)

$$D = \begin{pmatrix} s \\ c \\ l \end{pmatrix}.$$

The gauge fields of shsc$(1|3)$ are of the form

$$\begin{aligned} \omega_\mu &= \sum i^{-|2s|_2} \omega_{\mu,}^{(s,c)\alpha(2l)} T_{\alpha(2l)}^{(s,c)}, \qquad \mu = 0, 1, 2, \\ \omega_\mu^+ &= -\omega_\mu, \end{aligned} \tag{C.7}$$

where the fields and the generators have the same statistics, and which are compatible with a spin.

The curvatures shsc$(1|3)$ are obtained with the help of (C.3), (C.6), taking into account the statistics of fields and generators,

$$\begin{aligned} R_{\mu\nu,\alpha(2l)}^{(s,c)} &= \partial_{[\mu} \omega_{\nu],\alpha(2l)}^{(s,c)} + \sum i^{s'+s''-s-1} \delta(c'+c''-c) \\ &\quad \times \delta(2p-l'-l''+l)\, \delta(2q-l'+l''-l)\, \delta(2t-l''+l'-l) \\ &\quad \times \delta(|4s's''+s'+s''-s+1|_2) \\ &\quad \times \begin{pmatrix} s' & s'' & s \\ c' & c'' & c \\ l' & l'' & l \end{pmatrix} \omega_{\mu,\alpha(2q)\gamma(2p)}^{(s',c')} \omega_{\nu,\alpha(2t)}^{(s'',c'')\gamma(2p)}. \end{aligned} \tag{C.8}$$

D. *The Superconformal Basis*

In this appendix we shall perform the calculations, leading from (4.5) and (6.7) to the formulas (4.6) and (6.9) for the conformal bases in $hsc^\infty(4)$ and $shsc^\infty(4|1)$.

Using the definition of the differential operator $\tilde{P}_{\alpha\beta}$ (3.11a), it is not difficult to obtain that

$$i^{(\sigma-c)} \underbrace{\tilde{P}_{\gamma\delta} \cdots \tilde{P}_{\gamma\delta}}_{\sigma-c} (\bar{a}_{\rho(\sigma-u)} a_{\xi(\sigma+u)})$$

$$= \sum_{\sigma_1+\sigma_2=\sigma} \frac{(-1)^{\sigma_2-(c+u)/2}(\sigma-u)!\,(\sigma+u)!\,(\sigma-c)!}{\left(\sigma_1+\frac{c-u}{2}\right)!\left(\sigma_1-\frac{c-u}{2}\right)!\left(\sigma_2+\frac{c+u}{2}\right)!\left(\sigma_2-\frac{c+u}{2}\right)!}$$

$$\times \varepsilon_{\gamma(\sigma_2-(c+u)/2),\,\rho(\sigma_2-(c+u)/2)}\, \varepsilon_{\delta(\sigma_1-(c-u)/2),\,\xi(\sigma_1-(c-u)/2)}\, \bar{a}_{\rho(\sigma_1+(c-u)/2)}$$

$$\times a_{\gamma(\sigma_1-(c-u)/2)}\, \bar{a}_{\delta(\sigma_2-(c+u)/2)}\, a_{\xi(\sigma_2+(c+u)/2)}, \qquad \sigma_1, \sigma_2 = 0, \tfrac{1}{2}, 1, \ldots \tag{D.1}$$

Using the relation

$$\bar{C}_{\alpha(2l),}{}^{\gamma(2l'),\rho(2l'')}\, \varepsilon_{\gamma(k),\rho(k)}$$

$$= \sqrt{\frac{(2l'-k)!\,(2l''-k)!\,(l'+l''-l)!\,(l+l'+l''+1)!}{(2l')!\,(2l'')!\,(l'+l''-l-k)!\,(l+l'+l''-k+1)!}}$$

$$\times \bar{C}_{\alpha(2l),}{}^{\gamma(2l'-k),\rho(2l''-k)}, \tag{D.2}$$

which follows from

$$\bar{C}_{\alpha(2l),}{}^{\gamma(2l'),\rho(2l'')}\varepsilon_{\gamma\rho} = \sqrt{\frac{(l'+l''-l)(l+l'+l''+1)}{(2l')(2l'')}}$$

$$\times \bar{C}_{\alpha(2l),}{}^{\gamma(2l'-1),\rho(2l''-1)}, \tag{D.3}$$

the expression (4.5) can be rewritten modulo a constant factor as

$${}^{(n)}T^{(s,c)}_{\alpha(2l),\beta(2j)} \sim i^{c-s-1} \sum_{s_1+s_2=s} (-1)^{s_2-c/2}$$

$$\times \sqrt{\frac{(s+l+j+2)!\,(s-l-j)!\,(s+l-j+1)!\,(s+j-l+1)!}{(2s+2)!\,(s_2+j+1)!\,(s_2-j)!\,(s_1+l+1)!\,(s_1-l)!}}\; T^{(s_1,s_2,\,c/2,c/2)}_{\alpha(2l),\beta(2j)}$$

$$\times (\bar{a}_\alpha a^\alpha + \bar{a}^\beta a_\beta)^n. \tag{D.4}$$

Performing the calculations in (D.4) and noticing, that, as it follows from the definition (4.7) and from the expressions for spinorial C–G coefficients (B.3),

$$(\bar{a}_\alpha a^\alpha)^k(\bar{a}^{\dot\beta}a_{\dot\beta})^{n-k}T^{(s_1,s_2,c/2,c/2)}_{\alpha(2l),\dot\beta(2j)}$$
$$=\sqrt{\frac{(s_1+k-l)!(s_1+l+k+1)!(s_2+n-k-j)!(s_2+j+n-k+1)}{(s_1-l)!(s_1+l+1)!(s_2-j)!(s_2+j+1)!}}$$
$$\times T^{(s_1+k,s_2+n-k,c/2,c/2)}_{\alpha(2l),\dot\beta(2j)}, \tag{D.5}$$

we obtain

$$ {}^{(n)}T^{(s,c)}_{\alpha(2l),\dot\beta(2j)} \sim i^{c-s-1}\sum_{s_1+s_2=s+n}(-1)^{s_2-n-c/2}\left[n!\sqrt{\frac{(s+l-j+1)!(s+j-l+1)!}{(2s+2)!}}\right.$$
$$\left.\times\sum_k\frac{(-1)^k\sqrt{(s_1-l)!(s_1+l+1)!(s_2-j)!(s_2+j+1)!(s-l-j)!(s+l+j+2)!}}{k!(n-k)!(s_1-k-l)!(s_1-k+l+1)!(s_2-n+k-j)!(s_2-n+k+j+1)!}\right]$$
$$\times T^{(s_1,s_2,c/2,c/2)}_{\alpha(2l),\dot\beta(2j)}. \tag{D.6}$$

It is not difficult to notice, that the expression in the square brackets in (D.6) is equal to

$$\sqrt{\frac{(2s+n+3)!n!}{(2s+3)!}}\,C^{(s+n-l+j+1)/2,(s+n+l-j+1)/2,s+1}_{(s_2-s_1+l+j+1)/2,(s_1-s_2+l+j+1)/2,l+j+1}, \tag{D.7}$$

where $C$ are the usual C–G coefficients. The relations (D.6), (D.7) prove the formula (4.6) for a conformal basis. The normalization of basis vectors is chosen in such a way that the relation (4.10) holds. Formula (6.9) is proved analogously.

E. *Matrices of Transformations* (5.12), (7.13)

In this appendix we calculate the matrix $C_{m,k}$ of a transformation from the basis

$$ {}^{(m)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}=\tilde{\tilde{T}}_{A_1\cdots A_n,}{}^{B_1\cdots B_n}(T)^m \tag{E.1}$$

to the basis

$$ {}^{(k)}\tilde{\tilde{T}}_{A_1\cdots A_n,}{}^{B_1\cdots B_n}=\tilde{\tilde{T}}_{A_1\cdots A_n,}{}^{B_1\cdots B_n}*\underbrace{T*\cdots*T}_{k}. \tag{E.2}$$

This transformation has the form

$$ {}^{(m)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}=\sum_{k=0}^{m}C^{2n+M-N}_{m,k}\,{}^{(k)}\tilde{\tilde{T}}_{A_1\cdots A_n,}{}^{B_1\cdots B_n}. \tag{E.3}$$

Multiplying both sides of (E.3) on $T$ ($*$-product) and noticing that

$$T*{}^{(m)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}$$
$$={}^{(m+1)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}-\frac{\partial_l}{\partial\bar a}\cdot\frac{\partial_l}{\partial a}{}^{(m)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}$$
$$={}^{(m+1)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}-m(m+M-N+2n-1)\,{}^{(m-1)}T_{A_1\cdots A_n,}{}^{B_1\cdots B_n}, \tag{E.4}$$

we obtain the recurrent relation for $C_{m,k}$,

$$C_{m,k-1}^{2n+M-N} = C_{m+1,k}^{2n+M-N} - m(m+M-N+2n-1)\,C_{m-1,k}^{2n+M-N}. \tag{E.5}$$

The number M–N in (E.4, 5) appears due to the equality

$$\mathrm{str}(\delta_B^A) = \sum_A (-1)^{\varepsilon_A} \delta_A^A = M-N, \qquad A, B = 1, ..., M+N.$$

The initial conditions in Eq. (E.5) are

$$C_{0,k}^{2n+M-N} = \delta_{0,k}, \qquad ({}^{(0)}T_{A_1 \cdots A_n,}{}^{B_1 \cdots B_n} = {}^{(0)}\tilde{\tilde{T}}_{A_1 \cdots A_n,}{}^{B_1 \cdots B_n}). \tag{E.6}$$

The recurrent relation is easily solved by using a generating function

$$G^\alpha(Z, X) = \sum_{m=0}^{\infty} \frac{Z^m}{m!} \sum_{k=0}^{n} C_{m,k}^\alpha X^k, \qquad \alpha = M-N+2n. \tag{E.7}$$

Then Eq. (E.5) and initial conditions (E.6) are expressed in a differential equation

$$\frac{\partial}{\partial Z} G^\alpha(Z, X) = \frac{X+\alpha Z}{1-Z^2} G^\alpha(Z, X), \qquad G^\alpha(0, X) = 1, \tag{E.8}$$

which can be immediately integrated

$$G^\alpha(Z, X) = \left(\frac{1+Z}{1-Z}\right)^{X/2} (1-Z^2)^{-\alpha/2} \tag{E.9}$$

in a range $|Z| < 1$, $Z \in \mathbb{C}$. Thus the matrix elements of the transformations (E.3) are given by

$$C_{m,k}^{2n+M-N} = \frac{m!}{2^k k!} \oint_0 \frac{dZ}{2\pi i} \frac{\ln^k\left(\frac{1+Z}{1-Z}\right)}{(1-Z^2)^{n+(M-N)/2}\, Z^{m+1}}, \tag{E.10}$$

We shall need a particular value of $C_{m,k}$ at $k=0$,

$$C_{2m,0}^{2n+M-N} = (2m)! \oint_0 \frac{dZ}{2\pi i} \frac{(1-Z^2)^{(N-M)/2-n}}{Z^{2m+1}} = \frac{(2m)!}{m!} \left(n + \frac{M-N}{2}\right)_m, \tag{E.11a}$$

$$C_{2m+1,0}^{2n+M-N} = 0,$$

$$(a)_m = \prod_{k=0}^{m-1} (a+k), \; m > 0 \qquad \text{and} \qquad (a)_0 = 1. \tag{E.11b}$$

The matrix of transformation in (5.12) has the form

$$C_{n,m}(s)=\sqrt{\frac{(2s+3)!}{(2s+n+3)!\,n!}}\,C_{n,m}^{2s+4}, \tag{E.12}$$

as it follows from (see (4.6), (E.1), (D.6), (D.7))

$${}^{(n)}T^{(s,c)}_{\alpha(2l),\dot\beta(2j)}=\sqrt{\frac{(2s+3)!}{n!\,(2s+n+3)!}}\;{}^{(0)}T^{(s,c)}_{\alpha(2l),\dot\beta(2j)}\,(T)^n. \tag{E.13}$$

When obtaining the structure coefficients of an algebra hsc(4) (5.17), the particular value (E.12) is used:

$$C_{2m,0}(s)=\sqrt{\frac{(2s+3)!\,(2m)!}{(2s+2m+3)!}}\,\frac{(s+m+1)!}{m!\,(s+1)!}. \tag{E.14}$$

In the case of a superalgebra shsc(4 | 1), the matrix of transformation (7.13) has the form

$$\bar C_{n,m}(s)=\sqrt{\frac{(2s+2)!}{(2s+n+2)!\,n!}}\,C_{n,m}^{2s+3}, \tag{E.15}$$

as it follows from

$${}^{(n)}T^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)}=\sqrt{\frac{(2s+2)!}{(2s+n+2)!\,n!}}\;{}^{(0)}T^{(s,\sigma,c,u)}_{\alpha(2l),\dot\beta(2j)}\,(\bar a\cdot a+\alpha^+\alpha)^n. \tag{E.16}$$

The equality (E.16) is easily proven with the help of (6.11), (6.12). The particular values of the coefficients (E.15), necessary for the calculation of the structure coefficients of shsc(4 | 1) have the form

$$\bar C_{2m,0}(s)=\sqrt{\frac{(2s+2)!\,(2m)!}{(2s+2m+2)!}}\,\frac{(s+3/2)_m}{m!}. \tag{E.17}$$

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