Modern Physics Letters A Vol. 4, No. 24 (1989) 2363–2375

CONFORMAL SUPERALGEBRAS OF HIGHER SPINS

E. S. FRADKIN and V. Ya. LINETSKY
P. N. Lebedev Physical Institute of the USSR, Academy of Sciences, Leninsky prospect 53, 117924 Moscow, USSR

Received 17 August 1989

Infinite-dimensional conformal higher spin superalgebras are constructed. Based on the superalgebra in three dimensions, an explicit expression for the effective action is found. In four dimensions, the curvatures of higher spin conformal superalgebras are obtained.

1. Introduction

In Refs. 1–8 infinite-dimensional Lie superalgebras were constructed, generalizing the usual supergravity superalgebras in anti-de Sitter space adS_4. On the basis of these superalgebras in Refs. 1–8 the theories of interacting massless higher spins in adS_4 are developed. In these papers, it was established that the interaction of higher spins among themselves and with gravity is nonanalytical with respect to a cosmological constant and does not permit to pass to a flat limit.

In Refs. 14–17 it was shown that in the flat space-time, the consistent interactions of higher spins with gravity does not exist. However, in these papers the case of usual Einstein gravity was considered.

A different situation is possibly realized in the framework of conformally invariant approach. In the conformally invariant theory, the higher spins are described by kinetic terms with higher derivatives[13] and that, in principle, gives the opportunity of constructing the consistent interaction of higher spins with Weyl gravity.

Such theories are interesting from several points of view. Firstly, as it is known from the experience with supergravity that knowledge of the conformally-invariant theory of higher spins helps in construction of the necessary set of auxiliary fields, which are needed for the closure of the gauge algebra in theory of Refs. 1–8, and knowing the complete set of auxiliary fields, one could advance to construct a full interaction. Secondly, the conformal invariance poses strong restrictions on the possible interactions and there exists hope of constructing a full lagrangian. The extended superconformal theories with $N \geq 4$ are especially interesting. For example, in the theory with $N = 5$, the gauge group contains the group of the "Grand Unification" SU(5), which is suitable for the description of low-energy physics. Without higher spins, the construction of such a theory appeared to be impossible. Thirdly, the possibilities of a spontaneous breaking of conformal symmetry in such theories are very attractive. One of the possible scenarios can lead to the appearance of massive higher spins and to the reduction of a conformal gravity to Einstein gravity. The resulting theory would describe the interaction of massive higher spins with gravity analogously to a superstring theory.

Another possible scenario of a spontaneous symmetry breaking could lead to the appearance of a cosmological constant and to a theory of higher spins in adS_4.[1-8] The first step towards a construction of a conformally invariant theory of higher spins is a construction of infinite-dimensional superalgebras containing the usual conformal superalgebras as their subalgebra. These superalgebras could be the possible candidates for the role of the superconformal symmetry of higher spins, which generalizes the gauge symmetry of conformal supergravity.

In Refs. 18, 19 we analyze the case of the three-dimensional space-time. We construct the infinite-dimensional superalgebras shsc($N|3$) (super higher spin conformal), which generalize the superalgebras of conformal supergravity in $D = 3 - \mathrm{osp}(N|4)$. The gauge fields of shsc($N|3$) generalize the fields of conformal supergravity in $D = 3$. In this case (see below), to construct the complete gauge-invariant action and to write a generating functional for the Green's functions seem possible. However, in three dimensions the on-mass shell dynamics is trivial.

Here we begin to solve the problem of construction of the four-dimensional higher spin superconformal theory. We construct a whole series of infinite-dimensional superalgebras, generalizing the conformal superalgebra SU(2, 2|1). These superalgebras $\mathrm{shsc}^{(n)}(4|1)$, $n = 1, 2, \ldots, \infty$ are the possible candidates for the role of the conformal symmetry of higher spins. We obtain explicit expressions for the gauge curvatures of these superalgebras. The superconformal gauge theory of higher spins based on these results will be described in a separate paper.

2. The Method of Construction Conformal Superalgebras of Higher Spins

The construction of a conformal superalgebra of higher spins proceeds through the following stages:

(1) A choice of a convenient operatorial realization of a finite-dimensional subalgebra[9] with Heisenberg generating elements $\hat{a}_A$

$$[\hat{a}_A, \hat{a}_B\} = \mathbb{C}_{A,B}. \tag{1}$$

(2) A construction of an associative algebra of polynomials[4] with respect to generating elements, chosen in the abstract.

Unlike the case of adS_4-superalgebras,[4] we must introduce in this algebra a special superconformal basis, in which all generators have definite conformal weight and lorentzian spinorial structure. Later with the help of the formulae for the multiplication of Weyl symbols A of the operators $\hat{A}$ (see Refs. 4, 10)

$$A * B = A \exp(\overleftrightarrow{\Delta}) B, \tag{2}$$

where

$$2\overleftrightarrow{\Delta} = \frac{\overleftarrow{\partial}}{\partial a_A} \mathbf{C}_{A,B} \frac{\overrightarrow{\partial}}{\partial a_B}; \tag{3}$$

the structure coefficients of an associative algebra are calculated. The structure of a Lie superalgebra is introduced by fixing the grassmanian parity of generators and by defining the supercommutator

$$[A,B\} = A * B - (-1)^{\varepsilon(A)\varepsilon(B)} B * A. \tag{4}$$

The conformal superalgebras in three- and four-dimensional space-time are osp$(N|4)$ and SU$(2,2|N)$, respectively. To realize the osp$(N|4)$ superalgebra, a suitable choice of generating elements is

$$[\hat{a}_\alpha, \hat{b}_\beta] = i\varepsilon_{\alpha\beta}, \hat{a}^\dagger_\alpha = \hat{a}_\alpha, \hat{b}^\dagger_\alpha = \hat{b}_\alpha, \tag{5a}$$

$$\{\hat{\psi}_i, \hat{\psi}_j\} = \delta_{ij}, \hat{\psi}^\dagger_i = \hat{\psi}_i, i = 1, \ldots, N, \tag{5b}$$

where α, β are the two-component spinorial indices in three dimensions. All operators quadratic in $(\hat{a}, \hat{b}, \hat{\psi})$ furnish osp$(N|4)$. The Weyl symbols of all even orders polynomial operators in this generating elements form an associative algebra aq$^E(N|4; \mathbf{C})$ [2] with multiplication (2). The structure of a Lie superalgebra shsc$(N|3)$ is introduced by the supercommutator (4), where the Grassmann parity is $A(a, b, \psi) = A(-a, -b, \psi)(-1)^{\varepsilon(A)}$ and the antihermitean condition $A^\dagger = -A$.

To understand the SU$(2, 2|N)$ superalgebra, a convenient choice of generating elements is

$$[a^\alpha, \bar{a}_\beta] = \delta^\alpha_\beta, [a_{\dot{\alpha}}, \bar{a}^{\dot{\beta}}] = \delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{6a}$$

$$(a^\alpha)^\dagger = \bar{a}^{\dot{\alpha}}, (\bar{a}_\beta)^\dagger = a_{\dot{\beta}}, (a_{\dot{\beta}})^\dagger = \bar{a}_\beta, (\bar{a}^{\dot{\beta}})^\dagger = a^\beta, \tag{6b}$$

$$\{\alpha_i, \alpha^\dagger_j\} = \delta_{ij}, (\alpha_i)^\dagger = \alpha^\dagger_i, (\alpha^\dagger_i)^\dagger = \alpha_i, i, j = 1, \ldots, N. \tag{6c}$$

All operators[a] quadratic in $(a, \bar{a}, \alpha, \alpha^\dagger)$ furnish osp$(2N|8)$, whereas those commuting with the "particle number" operator

[a] Both the operator and their symbols are denoted by the same letters. Let us mention that the supertwistors $Z = (a^\alpha, a_{\dot{\beta}}, \alpha)$ (and the dual $\bar{Z} = (\bar{a}_\alpha, \bar{a}^{\dot{\beta}}, \alpha^\dagger)$) are natural for the construction of conformal superalgebras.

$$T = \bar{a}_\alpha a^\alpha + \bar{a}^{\dot\beta} a_{\dot\beta} + \alpha^\dagger \alpha \tag{7}$$

(excluding the operator (7) itself) form a subalgebra of osp$(2N|8)$ which is SU$(2,2|N)$.

Our next step is to consider of higher order polynomials in the generating elements. The associative algebra of those polynomials in the generating elements (6) that commute with the "particle number" operator is called aqpc$(2,2|N;\mathbb{C})$ (associative quantum particle conversation). The commutation relations (4) with the Grassmann parity

$$A(-a, -\bar{a}, \alpha, \alpha^\dagger) = (-1)^{\varepsilon(A)} A(a, \bar{a}, \alpha, \alpha^\dagger) \tag{8a}$$

and antihermitean condition

$$A^\dagger = -A \tag{8b}$$

endow aqpc with a Lie superalgebra structure, which we denote as iU$(2,2|N)$ (infinite dimensional unitary).

To construct a gauge theory, however, it is necessary to introduce in iU$(2,2|N)$ a particular superconformal basis in which iU$(2,2|N)$ would be explicitly decomposed into SU$(2,2|N)$ irreducible representations.

To construct this basis, the standard technique of the representation theory of Lie algebras is to be used. One finds out all the highest vectors with maximal conformal weight and acts on them with operators which lower conformal weight. Then an irreducible spinor basis is obtained through the use of spinorial Clebsch-Gordan Sl$(2;\mathbb{C})$ coefficients. The curvatures are calculated in the superconformal basis with the help of the known structure coefficients for the three-dimensional conformal superalgebra shsc$(N|3)$.

Factoring out the iU$(2,2|N)$ to its centre, generated by powers of the "particle number" operator (1.7), one obtains the superalgebra shsc$^\infty(4|N)$.

3. The Three-Dimensional Theory of Higher Spins

Following the operatorial method of constructing conformal superalgebras of higher spins, we obtained in Ref. 18 the superalgebra in three-dimensional space-time. With the help of this result, we obtain the complete quantum conformal theory for all higher spins.

The gauge fields of this superalgebra shsc$(N|3)$ have a form (our notations are explained in the Appendix)

$$\omega_\mu = \sum_{s,c,k,l} i^{-|2s|_2}\, \omega_{\mu,(s,c)}^{i(k),\alpha(2l)}\, T_{i(k),\alpha(2l)}^{(s,c)}, \tag{9}$$

where the generators T are the polynomials in generating elements (5). The summation parameters in (9) take the values

$$s = 0, \frac{1}{2}, 1, \ldots, \infty; c = -s, -s+1, \ldots, s;$$

$$l = |c|, |c| + 1, \ldots, s; k = 0, 1, \ldots, N \text{ and } k+2s \text{ is even,}$$

s is a spin of conformal multiplet of generators[b], c is the conformal weight of fields and generators, $2l$ and k are the numbers of a Lorentz and internal indices, respectively. The fields and generators with integer (half-integer) spin s are Grassmann even (odd).

The Hermitean conjugation has a form

$$\omega_\mu^\dagger = -\omega_\mu, (\omega_{\mu,(s,c)}^{i(k),\alpha(2l)})^\dagger = (-1)^{k(k-1)/2}\, \omega_{\mu,(s,c)}^{i(k),\alpha(2l)}. \tag{10}$$

The curvature of shsc$(N|3)$, defined by the general formula

$$\mathscr{R}_{\mu\nu} = \partial_{[\mu}\omega_{\nu]} + [\omega_\mu, \omega_\nu] \tag{11}$$

have a form[18]

$$\begin{aligned}\mathscr{R}^{(s,c)}_{\mu\nu,\, i(k),\alpha(2l)} = {} & \partial_{[\mu}\omega^{(s,c)}_{\nu],\, i(k),\alpha(2l)} + \sum i^{s'+s''-s+r-|r|_2-1}\frac{k!}{u!v!r!}\,\delta(c'+c''-c)\\ & \times\delta(k-u-v)\delta(2p-l'-l''+l)\delta(2q-l-l'+l'')\\ & \times\delta(2t-l-l''+l')\delta(|4s's''+s'+s''-s+uv+r(u+v)+1|_2)\\ & \times\begin{pmatrix} s' & s'' & s\\ c' & c'' & c\\ l' & l'' & l\end{pmatrix}\omega^{(s',c')}_{\mu,\, i(u)j(r),\alpha(2q)\gamma(2p)}\,\omega^{(s'',c'')j(r)}_{\nu,\, i(v),\alpha(2t)}\,\gamma(2p),\end{aligned} \tag{12}$$

where the matrix of structure coefficients is given by the relations[c]

$$\begin{pmatrix} s & s' & s''\\ c & c' & c''\\ l & l' & l''\end{pmatrix} = \sqrt{\frac{(2l+1)!(2l'+1)!(2l''+1)!}{(l+l'-l'')!(l+l''-l')!(l'+l''-l)!(l+l'+l''+1)!}}$$

[b] The corresponding gauge fields has a spin $s+1$ due to a vector index.

[c] In the expression, (13) is the usual elements theory of the angular momentum[20]–$9j$–coefficients, Wigner d-functions and triangle coefficients Δ. The summation indices in (13) $k, k', k'' = -l, -l+1, \ldots, l$. There is a slip of pen in Eq. (15) of Ref. 18, the correct expression is Eq. (13) of this article.

$$\times \sum_{k,k',k''} (-1)^{1/2(s+s'-s''-k-k'+k'')}$$

$$\times \frac{d^{l}_{c,k}\left(\frac{\pi}{2}\right) d^{l'}_{c',k'}\left(\frac{\pi}{2}\right) d^{l''}_{c'',k''}\left(\frac{\pi}{2}\right)}{\sqrt{\Delta\left(\frac{s+k}{2},\frac{s'+k'}{2},\frac{s''+k''}{2}\right)\Delta\left(\frac{s-k}{2},\frac{s'-k'}{2},\frac{s''-k''}{2}\right)}}$$

$$\times \begin{Bmatrix} \frac{s+k}{2}, & \frac{s-k}{2}, & l \\ \frac{s'+k'}{2}, & \frac{s'-k'}{2}, & l' \\ \frac{s''+k''}{2}, & \frac{s''-k''}{2}, & l'' \end{Bmatrix}. \tag{13}$$

It is assumed that the coefficients (13) are equal to zero, if at least one of the following conditions is not fulfilled

$$\begin{gathered} s'' \in \{|s-s'|,\dots,s+s'\};\, l'' \in \{|l-l'|,\dots,l+l'\}; \\ c+c'-c'' = 0;\, l \in \{|c|,\dots,s\};\, l' \in \{|c'|,\dots,s'\}; \\ l'' \in \{|c''|,\dots,s''\};\, |c| \leqslant s;\, |c'| \leqslant s';\, |c''| \leqslant s'', \end{gathered} \tag{14}$$

and all numbers in each column in (13) are simultaneously integer or half-integer.

The formal summation range in (12) is

$$\begin{gathered} s', s'', l', l'', p, q, t = 0, \frac{1}{2}, 1, \dots, \infty; \\ c', c'' = -\infty, \dots, -\frac{1}{2}, 0, \frac{1}{2}, \dots, \infty;\, u, v, r = 0, 1, \dots, \infty. \end{gathered} \tag{15}$$

It is however additionally restricted, due to a definition of the coefficients (13) in (14), by a δ-functions, by the factorials in the denominator and by the antisymmetry with respect to internal indices.

The action of superconformal-invariant theory of higher spins in $D = 3$ can be written in a form of a Chern-Simons functional

$$\mathcal{S} = \int \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right),$$

$$\omega = \omega_\mu dx^\mu,\ \omega\wedge\omega = \frac{1}{2}(\omega_\mu * \omega_\nu - \omega_\nu * \omega_\mu)dx^\mu \wedge dx^\nu, \tag{16}$$

where the trace of the Weyl symbol of the operator is defined by the formula

$$\mathrm{Tr}\,(A(g)) = A(0), \tag{17}$$

where g is the symbol of generating elements (5). The equations of motion have a form

$$\mathcal{R}^{(s,c)}_{\mu\nu,\,i(k),\,\alpha(2l)} = 0. \tag{18}$$

This theory is a gauge-invariant theory of rank one and we perform the quantization of (16) according to the general rules of Refs. 10–12.

The generating functional of all Green functions of the superconformal theory in three-dimensional space-time has a form

$$\mathcal{Z} = \int \mathcal{D}\,\omega^A_\mu\, \mathcal{D}\,\pi_A\, \mathcal{D}\, C^A\, \mathcal{D}\,\bar{C}_A \exp\{i\,\mathcal{S}_{\mathrm{eff}}\}, \tag{19}$$

where

$$\mathcal{S}_{\mathrm{eff}} = \mathcal{S} + \int d^3x\, \mathrm{Tr}\left(\pi * \frac{\delta_l \Phi}{\delta \bar{C}} + \frac{\delta_r \Phi}{\delta \omega_\mu} * \mathcal{D}_\mu C + \frac{\delta_r \Phi}{\delta C} * C * C + \mathcal{T}_\mu * \omega^\mu\right), \tag{20}$$

$C = C^A\, T_A$, $\bar{C} = \bar{C}_A T^A$, $\mathcal{D}_\mu$ – shsc$(N|3)$ covariant derivative, C^A, $\bar{C}_A$ are the ghosts and antighosts, $T_A(T^A)$ – generators of shsc$(N|3)$, and the fermionic generator of the gauge conditions are

$$\Phi = \int d^3x\, \bar{C}_A \Phi^A(\omega, C, \bar{C}),\ \frac{\delta_{r,l}\Phi}{\delta\omega} = \frac{\delta_{r,l}\Phi}{\delta\omega_A} T_A \quad \text{etc.} \tag{21}$$

One can convince himself, that as usually for Chern-Simons theory in the three-dimensional space-time, the dynamics on the mass-shell is trivial.

4. The Conformal Superalgebras of Higher Spins in Four-Dimensional Space-time

Our next problem is to obtain the conformal superalgebra for higher spins in four-dimensional space-time. Following the programme of Sec. 2 we obtain the structure constants of higher spin conformal superalgebras and the explicit expressions of the gauge fields and curvatures.

The gauge fields of the superalgebra $\mathrm{shsc}^{\infty}(4|1)$ have the form

$$\omega_{\mu} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \sum_{\substack{\mathscr{s},c,u \\ l,j}} {}^{(n)}\omega_{\mu,\alpha(2l),\dot{\beta}(2j)}^{(s,\mathscr{s},c,u)}\, {}^{(n)}T_{(s,\mathscr{s},c,u)}^{\alpha(2l),\dot{\beta}(2j)}\, i^{-|2\mathscr{s}|_2}, \mu = 0, 1, 2, 3, \tag{22}$$

where for the generators ${}^{(n)}T^{(s,\mathscr{s},c,u)}_{\alpha(2l),\dot{\beta}(2j)}$ s-is a highest spin of a conformal supermultiplet $s = 1, 2, \ldots, \infty$; $\mathscr{s}$ – is the spin of a given conformal multiplet in a supermultiplet $\mathscr{s} = s, s-1/2, s-1$; $c \sim$ is the conformal weight of a field $c = -s, -s+1, \ldots, s$; u – is the chiral weight of a field $u = -1/2, 0, 1/2$ ($u+\mathscr{s}$ – integer); $[u, T^{(u)}] = 3/2\ iu\ T^{(u)}$; (l, j)–is the Lorenzian signature of a field, where $l, j = 0, 1/2, 1, \ldots, l \geq |(c-u)/2|, j \geq |(c+u)/2|, l+j \leq \mathscr{s}, l+(c-u)/2$ and $j+(c+u)/2$ are integer. The index $n+1 = 1, \ldots, \infty$ numerates the identical supermultiplets. The gauge fields (22) generalize the gauge fields of conformal supergravity to the case of higher spins.[d] It contains an infinite number of conformal supermultiplets of every spin (the spin of a field $\omega_{\mu}^{(s,\mathscr{s},c,u)}$ is $\mathscr{s}+1$ due to a vector index). The grassmanian parity of fields and generators is 0 (or 1) for integer (or half-integer) spin $\mathscr{s}$. The curvatures of the superalgebra $\mathrm{shsc}^{\infty}(4|1)$, defined by the general formula (11), have the form

$$\begin{aligned}
{}^{(n)}\mathscr{R}^{(s,\mathscr{s},c,u)}_{\mu\nu,\alpha(2l),\dot{\beta}(2j)} = {} & \partial_{[\mu}^{(n)}\omega^{(s,\mathscr{s},c,u)}_{\nu],\alpha(2l),\dot{\beta}(2j)} \\
& + \sum i^{\mathscr{s}'+\mathscr{s}''-\mathscr{s}-1}\,\delta(c'+c''-c)\delta(u'+u''-u) \\
& \times \delta(2t-l'+l''-l)\delta(2m-l'-l''+l)\delta(2r-l''+l'-l) \\
& \times \delta(2p-j'-j''+j)\delta(2q-j''-j+j')\delta(2k+j''-j-j')
\end{aligned}$$

[d] The conformal supergravity fields are: $(\omega_{\mu,\alpha,\dot{\beta}}^{(1,1,1,0)}; \omega_{\mu,\alpha(2)}^{(1,1,0,0)}; \omega_{\mu,\dot{\beta}(2)}^{(1,1,0,0)}; \omega_{\mu}^{(1,1,0,0)}; \omega_{\mu,\alpha,\dot{\beta}}^{(1,1,-1,0)}; \omega_{\mu,\dot{\alpha}}^{(1,1/2,1/2,1/2)}; \omega_{\mu,\alpha}^{(1,1/2,1/2,1/2,-1/2)}; \omega_{\mu,\alpha}^{(1,1/2,-1/2,1/2)}; \omega_{\mu,\dot{\alpha}}^{(1,1/2,-1/2,-1/2)}; \omega_{\mu}^{(1,0,0,0)}) \sim (f_{\mu,\alpha\dot{\beta}}; \omega_{\mu,\alpha(2)}; \omega_{\mu,\dot{\beta}(2)}; b_{\mu}; l_{\mu,\alpha\dot{\beta}}; \phi_{\mu,\dot{\alpha}}; \phi_{\mu,\alpha}; \psi_{\mu,\alpha}; \psi_{\mu,\dot{\alpha}}; A_{\mu})$.

$$\times\, \delta(|s+s'+s''+n+n'+n''+1|_2)$$

$$\times \begin{bmatrix} n' & s' & \mathscr{s}' & c' & u' & l' & j' \\ n'' & s'' & \mathscr{s}'' & c'' & u'' & l'' & j'' \\ n & s & \mathscr{s} & c & u & l & j \end{bmatrix}$$

$$\times\, {}^{(n')}\omega^{(s',\mathscr{s}',c',u')}_{\mu,\alpha(2t)\gamma(2m),\dot\beta(2k)\dot\rho(2p)}\; {}^{(n'')}\omega^{(s'',\mathscr{s}'',c'',u'')\gamma(2m)}_{\nu,\alpha(2r),\dot\beta(2q),\dot\rho(2p)}\,, \tag{23}$$

where the matrix of numerical structure coefficients is defined by the formula

$$\begin{bmatrix} n_1 & s_1 & \mathscr{s}_1 & c_1 & u_1 & l_1 & j_1 \\ n_2 & s_2 & \mathscr{s}_2 & c_2 & u_2 & l_2 & j_2 \\ n_3 & s_3 & \mathscr{s}_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix}$$

$$= \sum_{s_i'+n_i'=s_i+n_i}\; \sum_{\mathscr{s}_i'+\mathscr{s}_i''=\mathscr{s}_i+n_i'}\; \prod_{i=1}^{3} \{ C_{\mathscr{s}_i}(s_i,n_i,s_i',n_i')$$

$$\times\, C^{1/2\,(\mathscr{s}_i+n_i'-l_i+j_i+1),\; 1/2\,(\mathscr{s}_i+n_i'+l_i-j_i+1),\mathscr{s}_i+1}_{1/2\,(\mathscr{s}_i''-\mathscr{s}_i'+l_i+j_i+1),\; 1/2\,(\mathscr{s}_i'-\mathscr{s}_i''+l_i+j_i+1),\; l_i+j_i+1}\Big\} \times (-1)^{s_3'-s_3}\, A^{s_1'-\mathscr{s}_1,\, s_2'-\mathscr{s}_2,\, s_3'-\mathscr{s}_3}_{u_1,\, u_2,\, u_3}$$

$$\times \begin{pmatrix} \mathscr{s}_1' & \mathscr{s}_2' & \mathscr{s}_3' \\ \dfrac{c_1-u_1}{2} & \dfrac{c_2-u_2}{2} & \dfrac{c_3-u_3}{2} \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} \mathscr{s}_1'' & \mathscr{s}_2'' & \mathscr{s}_3'' \\ \dfrac{c_1+u_1}{2} & \dfrac{c_2+u_2}{2} & \dfrac{c_3+u_3}{2} \\ j_1 & j_2 & j_3 \end{pmatrix} \tag{24a}$$

Here

$$A^{0,0,0}_{0,0,0} = A^{1,1,0}_{0,0,0} = A^{0,1,1}_{0,0,0} = A^{1/2,1/2,0}_{1/2,-1/2,0} = A^{1,1/2,1/2}_{0,1/2,1/2} = A^{1/2,1/2,1}_{1/2,-1/2,0} = A^{0,1/2,1/2}_{0,1/2,1/2} = 1\,, \tag{24b}$$

$$A^{l_2,l_1,l_3}_{u_2,u_1,u_3} = A^{l_1,l_2,l_3}_{-u_1,-u_2,-u_3} = (-1)^{4l_1l_2+l_1+l_2-l_3} A^{l_1,l_2,l_3}_{u_1,u_2,u_3} \tag{24c}$$

and all the A-coefficients besides (24b) and obtainable from them with the help of the symmetry properties (24c) are equal to zero. The $C^{l,l',l''}_{m,m',m''}$ are the usual

Clebsch-Gordan coefficients and the non-zero coefficients $C_{\mathscr{s}}(s,n,s',n')$ are

$$C_{s-1/2}(n,s,n,s) = 1,\ C_s(n,s,n,s) = \sqrt{\frac{2s+n+3}{2s+3}},$$

$$C_s(n,s,n-1,s+1) = \sqrt{\frac{n}{2s+3}},$$

$$C_{s-1}(n,s,n+1,s-1) = \sqrt{\frac{n+1}{2s+1}},\ C_{s-1}(n,s,n,s) = \sqrt{\frac{2s+n+2}{2s+1}}. \quad (24d)$$

The structure coefficients of shsc$^\infty(4|1)$ (24) include the structure coefficients of shsc$(1|3)$ (13) and the coefficients of the transformation to the superconformal basis. We redefine them in such a way that they are equal to zero, if at least one of the following conditions is not fulfilled:

$$\begin{gathered} c_1+c_2=c_3,\ u_1+u_2=u_3,\ l_3\in\{|l_1-l_2|,\dots,l_1+l_2\}, \\ j_3\in\{|j_1-j_2|,\dots,j_1+j_2\},\ l_i+j_i\leqslant \mathscr{s}_i,\ l_i\geqslant\left|\frac{c_i-u_i}{2}\right|, \\ j_i\geqslant\left|\frac{c_i+u_i}{2}\right|,\ |c_i|\leqslant\mathscr{s}_i,\ \mathscr{s}_i\in\left\{s_i-1,\ s_i-\frac{1}{2},\ s_i\right\}, \\ s_3+n_3\in\{|s_1+n_1-s_2-n_2|,\dots,s_1+n_1+s_2+n_2\} \end{gathered} \quad (25)$$

and all numbers $\mathscr{s}_i$, c_i, u_i, l_i+j_i are simultaneously integer or half-integer for every $i=1,2,3$.

The formal summation range in (23) is

$$\begin{gathered} s',s''=1,2,\dots,\infty; \\ \mathscr{s}',\mathscr{s}'',l',l'',j',j'',t,m,k,p,q,r=0,\frac{1}{2},1,\dots,\infty; \\ u',u''=-\frac{1}{2},0,\frac{1}{2};\ c',c''=-\infty,\dots,-1,-\frac{1}{2},0,\frac{1}{2},\dots,\infty. \end{gathered} \quad (26)$$

The Hermitean conjugation acts at the fields (20) according to a formula

$$\omega_\mu^\dagger = -\omega_\mu,\ ({}^{(n)}\omega^{(s,\mathscr{s},c,u)}_{\mu,\alpha(2l),\dot\beta(2j)})^\dagger = {}^{(n)}\omega^{(s,\mathscr{s},c,-u)}_{\mu,\beta(2j),\dot\alpha(2l)}. \quad (27)$$

The derivation of formulae (23), (24) for the curvatures and the structure coefficients in the superalgebra is based on the operator representation of the generators of shsc$^\infty$ (4|1) is very lengthy and will be published in the complete form elsewhere. The curvature (23) generalize the curvatures of the conformal supergravity.[13] The superalgebra shsc$^\infty$(4|1) is not simple. In a special basis, connected with a basis (22) by the transformation of a form

$$^{(n)}\tilde{T}^{(s,\varepsilon,c,u)}_{\alpha(2l),\dot{\beta}(2j)} = \sum_{m=0}^{n} C_{n,m}(s)^{(m)}\, T^{(s,\varepsilon,c,u)}_{\alpha(2l),\dot{\beta}(2j)}, \tag{28}$$

the commutation relations of shsc$^\infty$ (4|1) take a form

$$[^{(n)}\tilde{T}, {}^{(m)}\tilde{T}\} = \sum_{k} {}^{(n+m+k)}\tilde{T}. \tag{29}$$

From these formulae we see that the sets of generators $^{(m)}\tilde{T}$ with $m \geqslant n$ for every fixed n form the family of the ideals, which are inserted into each other. The factor-algebras with respect to these ideals, are obtained by the identification

$$^{(m)}\tilde{T} \equiv 0, m \geqslant n, \tag{30}$$

we denote shsc$^{(n)}$ (4|1), $n = 1, 2, \ldots, \infty$.

Evidently, these superalgebras contain the conformal supermultiplets of all spins with the degeneracy equal to n. This family contains a simple superalgebra shsc$^{(1)}$(4|1), which we denote simply by shsc(4|1). This superalgebra contains the conformal supermultiplets of all spins only once and as its maximal finite-dimensional subalgebra contains an algebra of the usual conformal supergravity. The gauge fields and curvatures of shsc(4|1) are written as

$$\omega_\mu = \sum i^{-|2\varepsilon|_2}\, \omega^{\alpha(2l),\dot{\beta}(2j)}_{\mu,(s,\varepsilon c,u)}\, T^{(s,\varepsilon,c,u)}_{\alpha(2l),\dot{\beta}(2j)}, \tag{31}$$

$$\mathscr{R}^{(s,\varepsilon,c,u)}_{\mu\nu,\alpha(2l),\dot{\beta}(2j)}$$

$$= \partial_{[\mu}\omega^{(s,\varepsilon,c,u)}_{\nu],\alpha(2l),\dot{\beta}(2j)} + \sum \delta(c'+c''-c)\delta(u'+u''-u)\delta(2m-l'-l''+l)$$

$$\times\, \delta(2r-l''+l'-l)\delta(2t+l''-l'-l)\delta(2p-j-j''+j)$$

$$\times\, \delta(2q-j''-j+j')\delta(2k-j-j'+j'')\delta(|s+s'+s''+1|_2)$$

$$
\times\, i^{\mathscr{s}'+\mathscr{s}''-\mathscr{s}-1}\begin{bmatrix} s' & \mathscr{s}' & c' & u' & l' & j' \\ s'' & \mathscr{s}'' & c'' & u'' & l'' & j'' \\ s & \mathscr{s} & c & u & l & j \end{bmatrix}
$$

$$
\times\, \omega^{(s',\mathscr{s}',c',u')}_{\mu,\alpha(2t)\gamma(2m),\dot{\beta}(2k)\dot{\rho}(2p)}\, \omega^{(s'',\mathscr{s}'',c'',u'')\gamma(2m)}_{\gamma,\alpha(2r),\dot{\beta}(2q),\dot{\rho}(2p)}\,, \tag{32}
$$

where the new structure coefficients of factor algebra are connected with (24)

$$
\begin{bmatrix} s_1 & \mathscr{s}_1 & c_1 & u_1 & l_1 & j_1 \\ s_2 & \mathscr{s}_2 & c_2 & u_2 & l_2 & j_2 \\ s_3 & \mathscr{s}_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix} = \sum_m \sqrt{\frac{(2m)!(2s_3+2)!}{(2s_3+2m+2)!}}
$$

$$
\times \frac{\left(s_3+\frac{3}{2}\right)\left(s_3+\frac{5}{2}\right)\dots\left(s_3+m+\frac{1}{2}\right)}{m!}
$$

$$
\times \begin{bmatrix} 0 & s_1 & \mathscr{s}_1 & c_1 & u_1 & l_1 & j_1 \\ 0 & s_2 & \mathscr{s}_2 & c_2 & u_2 & l_2 & j_2 \\ 2m & s_3 & \mathscr{s}_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix}. \tag{33}
$$

Let us take note that the conformal basis in shsc$(4|1)$ is defined up to a common multiplier of the generators (31). The curvatures (32) straightforwardly generalize the curvatures of the usual conformal supergravity to the case of higher spins. The curvatures of conformal supergravity are contained in the formula (32), when $s = 1$.

The gauge field shsc$(4|1)$ contains all the fields with an integer spin twice (in the supermultiplets with highest spins s and $s+1$).

It can be shown, that all the results on the construction of superalgebras can be generalized to a case of the extended conformal supersymmetry of higher spins with arbitrary N. Let us mention that an important selection criterion for the proposed series of superalgebras is the possibility of their localization and of the construction of a selfconsistence complete theory. The solution of this problem would be done in the next article.

Appendix: Notations and Conventions

We follow the conventions of Refs. 1–8. The two-component spinorial indices are raised and lowered by means of $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon^{\alpha\beta}$, $\varepsilon_{12} = \varepsilon^{12} = 1$ as $A^\alpha = \varepsilon^{\alpha\beta}A_\beta$, $A_\beta = \varepsilon_{\alpha\beta}A^\alpha$ and analogously for dotted indices. The internal indices

$(i, j, k, \ldots)$ are raised and lowered by δ_{ij}, δ^{ij}. The metric has a signature $(+, -, -)$ and $(+, -, -, -)$ in three and four dimensions respectively.

A symmetrization (anti-symmetrization) is implied for any set of upper or lower spinorial (internal) indices denoted by alike letters. The usual summation convention is understood for each pair of a lower and an upper index denoted by the same letter. We often use the notations $\delta(n) = 1(0)$, $n = 0$ $(n \neq 0)$; $|n|_2 = \mathrm{mod}_2(n) = n - 2[n/2]$ ($[n/2]$ is the integer part of $n/2$).

References

1. M. A. Vasiliev and E. S. Fradkin, *Dokl. Acad. Nauk USSR* **29** (1986) 1100; *Zh. Eksp. Teor. Fiz.* **44** (1986) 484.
2. M. A. Vasiliev, *Fortschr. Phys.* **35** (1987) 741; *Nucl. Phys.* **B307** (1988) 319.
3. E. S. Fradkin and M. A. Vasiliev, *Ann. Phys.* **177** (1987) 63.
4. M. A. Vasiliev, *Fortschr. Phys.* **36** (1988) 33.
5. E. S. Fradkin and M. A. Vasiliev, *Phys. Lett.* **B189** (1987) 89.
6. E. S. Fradkin and M. A. Vasiliev, *Nucl. Phys.* **B291** (1987) 141.
7. E. S. Fradkin and M. A. Vasiliev, *Int. J. Mod. Phys.* **A3** (1988) 2983.
8. M. A. Vasiliev, *Phys. Lett.* **B209** (1988) 491.
9. M. Günaydin, *Lecture Note in Phys.* **180** (1983) 192.
10. I. A. Batalin and E. S. Fradkin, *Ann. Inst. Henri Poincaré* **49** (1988) 145.
11. E. S. Fradkin and T. E. Fradkina, *Phys. Lett.* **72B** (1978) 343.
12. I. A. Batalin and E. S. Fradkin, *La Rivista del Nuovo Cim.* **10** (1985) 1.
13. E. S. Fradkin and A. A. Tseytlin, *Phys. Rep.* **119** (1985) 233.
14. C. Aragone and S. Deser, *Phys. Lett.* **86B** (1979) 161.
15. S. Christensen and M. Duff, *Nucl. Phys.* **B154** (1979) 301.
16. F. A. Berends, J. W. van Holten, P. van Nieuwenhuizen, and B. de Wit, *J. Phys.* **A13** (1980) 1643.
17. B. de Wit and D. Z. Freedman, *Phys. Rev.* **D21** (1980) 358.
18. E. S. Fradkin and V. Ya. Linetsky, *Mod. Phys. Lett.* **A** (1989) 731.
19. E. S. Fradkin and V. Ya. Linetsky, *Ann. Phys.*, preprint.
20. B. C. Biedenharn and J. D. Louck, in *Angular momentum in quantum physics* (Addison-Wesley P. C., 1981).